POST MAIN SEQUENCE EVOLUTION OF SINGLE STARS[1]

×2064

Icko Iben, Jr.
University of Illinois, Champaign-Urbana, Illinois 61801

1 INTRODUCTION

This review is a sequel to a 1967 review by the same author (Iben 1967). Discussion is limited to the evolution of single stars in quasi-static phases, avoiding all discussion of binary systems and of final stellar states. For excellent reviews describing the theoretical evolution of binary systems, see Plavec (1968) and Paczynski (1971b). For discussions of final stellar states, see reviews on white dwarfs by Weidemann (1968) and Ostriker (1971), reviews on neutron stars by Cameron (1970) and Ruderman (1972), and a review on black holes by Misner et al (1973).

Section 2 of this review emphasizes that, despite the many years that we have studied the main sequence, we still may not understand it. Low-mass stars are the subject of section 3, where an attempt is made to show that most steps between the main sequence and white dwarf states are understood, at least qualitatively, and that even some modestly quantitative statements may be made about the internal properties of stars in old clusters. In section 4, evolution of intermediate-mass stars is discussed, with emphasis on core helium-burning stages and on the interior properties of stars with large carbon-oxygen cores in which electrons are degenerate. Finally, in section 5, progress in understanding more massive stars is briefly catalogued.

2 LINGERING UNCERTAINTIES IN MAIN SEQUENCE EVOLUTION

Since characteristics of post main sequence phases are influenced by prior events in the main sequence phase, it is appropriate to point out that many aspects of main sequence evolution are far from being understood.

2.1 *Sporadic Mixing and Initial Inhomogeneities in Low-Mass Stars*

Why does the Sun not produce neutrinos at calculated rates? Will the answer to the current dilemma significantly alter our picture of the evolution of the Sun and

[1] Supported in part by the National Science Foundation (GP-35863).

of other stars as well? Despite the many attempts to understand the difference between an experimentally determined neutrino capture rate (Davis 1972) and a rate estimated on the basis of stellar models (see a review by Bahcall & Sears 1972), no really convincing explanation of the discrepancy has yet emerged. One might argue that something is wrong with 1. solar models; 2. our theory of the neutrino or some other aspects of the nuclear and atomic physics that influences solar models, or 3. the experiment. Here, it is appropriate to address only the consequences of the first alternative.

Several recent attempts at solving the dilemma argue that the Sun may be undergoing a fluctuation on a thermal time scale (Sheldon 1969) because of an instability that may be a consequence of chemical inhomogeneities that build up on a nuclear-burning time scale (Dilke & Gough 1972, Ezer & Cameron 1972, Fowler 1972, 1973, Cameron 1973). Mixing over a large portion of the interior is assumed to be an end result of the instability (in particular, enhancing the abundance of He^3 near the center). It is not at all clear that the postulated mixing will be sufficiently effective (Rood 1972b) or even that the postulated instability is likely (Ulrich & Rood 1973, Defouw et al 1973, Schwarzschild & Härm 1973).

However, if sporadic mixing does occur in the Sun, there are at least two major ramifications that affect our understanding of other stars. 1. Since fuel being burned at the center is sporadically replaced by more fuel from regions further from the center, the time spent by a low mass star on or near the main sequence while burning hydrogen at the center will be increased over that expected in the absence of sporadic mixing. One of the most interesting effects would be an increase in the estimated ages of globular cluster stars (Shaviv & Salpeter 1971). 2. Following the exhaustion of hydrogen at the center, the path of a low-mass star might mimic that of more massive stars (in the core of which matter is continuously mixed by convection during the main sequence phase, even in the absence of the sporadic mixing that is being postulated). The net result is an increase in the age of the oldest galactic cluster that might be expected to exhibit a gap in the star distribution near the cluster turnoff point and a lowering of the minimum luminosity at which this gap can occur. It is interesting that, in the absence of some efficient mixing mechanism other than formal convection, it is difficult to account for the location of a possible gap in the cluster NGC 188 and to match the properties of the gap in the cluster M 67 (e.g., Aizenman et al 1969, Torres-Peimbert 1971).

Another suggestion for reducing neutrino fluxes requires that the solar interior has been spinning rapidly since birth. The accompanying reductions in central temperatures and densities lead to a reduced rate of nuclear burning when averaged over the past 4.7 billion years, and hence to a reduced current neutrino flux (Demarque et al 1973). However, spinning at the required rate would lead to meridional mixing and a consequent surface abundance distribution significantly affected by interior processing, a situation that is not supported by the facts (Iben 1969).

2.2 *Convective Overshoot in Stars of Intermediate Mass*

The extent of mixing in stars of intermediate mass that develop formal convective regions of finite size is also an uncertainty to be reckoned with. It is not impossible that overshoot beyond the edge of the formal core might significantly increase the size of the region over which matter is effectively mixed (e.g., Shaviv & Salpeter 1973). Several consequences are similar to those brought about by sporadic mixing in low-mass stars that do not develop a formal convective core in the absence of sporadic mixing. It has also been demonstrated (Maeder 1973) that the detailed path which a model star follows in the H-R diagram after the exhaustion of hydrogen at the center is influenced by the nature of the concentration gradients outside of the formal convective core. Maeder argues that better fits with observations can be achieved if overshoot is important.

2.3 *Semiconvection and Meridional Circulation in Massive Stars*

Two other types of mixing may play an important role in the evolution of massive stars. The first of these is "semi-convection" which, during the main sequence phase, is found to occur in a shell outside of the retreating formal convective zone. Whether or not the matter in this shell becomes fully convective during the overall contraction phase that terminates the main sequence phase depends on the choice of stability criterion, over which there still exists considerable controversy. The occurrence or non-occurrence of a fully convective shell influences where in the H-R diagram core helium burning will take place after the exhaustion of central hydrogen. (See, in particular, Simpson 1971 and Chiosi & Summa 1970.) Comparison between observations and the different theoretical consequences has been energetically pursued by many (e.g., Robertson 1973).

The second important mixing process is meridional circulation, which may, in a star that is rotating as rapidly as many upper main sequence stars are observed to rotate, mix matter throughout the star on a time scale comparable to or less than the main sequence lifetime. The consequences are an increase in main sequence lifetime and a change in surface composition that reflects results of nuclear processing in the deep interior. An order of magnitude estimate of quantitative changes in composition has recently been prepared by Paczynski (1973b). The influence of rotation on the evolution of internal state variables and on the evolution of bulk properties is reviewed by Fricke & Kippenhahn (1972).

3 EVOLUTION OF LOW-MASS STARS

3.1 *Quiescent Hydrogen Burning*

A number of studies published over the past seven years describe the dependence on composition of evolutionary characteristics up to the helium-burning phase (e.g., Demarque 1967, Iben & Faulkner 1968, Simoda & Iben 1968, 1970, Aizenman et al 1969, Iben & Rood 1970, Demarque et al 1970, Torres-Peimbert 1971, Hejlesen, P. M. 1972, Demarque & Mengel 1973, Rood 1973, and Hartwick & Vanden Berg 1973). In these studies it has been assumed that mixing processes other than formal convection may be neglected. Evolutionary tracks presented in these and in earlier papers have been used in attempts to date clusters and to establish estimates of the initial helium abundance for stars in these clusters (e.g., Simoda & Kimura 1968, Eggen & Sandage 1969, Sandage & Eggen 1969, Sandage

1970, and Simoda 1972). Age determinations are discussed extensively in the Proceedings of the IAU Colloquium No. 17 (Cayrel de Strobel & Delplace 1972). A review article by Larsson-Leander (1972) in these proceedings is particularly nice.

Of interest for quantitative comparisons with globular cluster stars are 1. cluster age as a function of the luminosity at the cluster turnoff point and as a function of composition parameters; 2. the mass in the helium core just prior to the helium flash as a function of composition and mass; and 3. the lifetime on the red giant branch as a function of input parameters.

The relationship between cluster age t_c (in units of 10^{10} yr), assumed helium abundance Y, "metal" or "heavy element" abundance Z, and turnoff luminosity L_{to} is approximately (Iben & Rood 1970)

$$\log t_c \approx 0.42 - 1.1 \log L_{to} + 0.59(0.3 - Y) - 0.14(\log Z + 3) \tag{1}$$

when $-4 \leqq \log Z \leqq -3$. A more general expression, valid for $-4 \leqq \log Z \leqq -2$, is given by Hartwick & Vanden Berg (1973).

At the onset of the helium flash, the mass in the hydrogen-exhausted core M_c depends on total stellar mass M and on abundance parameters Y and Z as (Rood 1973)

$$M_c \approx 0.475 + 0.23(0.3 - Y) - 0.01(\log Z + 3) + 0.035(0.8 - M) \tag{2}$$

where M_c and M are in solar units and $-4 \leqq \log Z \leqq -3$. For later use it is convenient to approximate this relationship by

$$\log M_c \approx -0.317 + 0.207(0.3 - Y) - 0.0091(\log Z + 3) \tag{2a}$$

where terms of order $(0.3 - Y)^2$ or higher have been neglected and M in (2) has been approximated by 0.6.

The lifetime t_{RG} (in units of 10^7 yr) of a giant evolving from luminosity L_{RG} (in solar units) up to the appropriate red giant tip where helium flashing occurs is related to M, Y, and Z by (Rood 1973)

$$\begin{aligned}\log t_{RG} \approx\ & 2.351 - 0.84 \log L_{RG} - 0.04(\log Z + 3) \\ & + 1.36 \log (1 - Y) - 0.27 \log M,\end{aligned} \tag{3}$$

and again $-4 \leqq \log Z \leqq -3$.

There are, of course, uncertainties in all three relationships beyond those occasioned by uncertainty as to the extent and consequences of mixing during the main sequence phase. For example, if postulated plasma neutrino losses do not occur, the mass of the helium core at the onset of the helium flash is smaller by about 0.03–$0.04 M_\odot$ than the value given by equations (2) and (2a). A reduction in core mass by $0.03 M_\odot$ means a reduction in the luminosity of a horizontal branch model by $\Delta \log L \sim 0.1$.

There seems to be no unanimity of opinion about whether or not a red giant's luminosity drops temporarily when the hydrogen-burning shell reaches a discontinuity in the hydrogen profile whose location is determined by the maximum inward extent of the convective envelope (Thomas 1967, Iben 1968a,b, Demarque & Mengel 1971, Demarque & Heasly 1971, Rood 1973). In all cases, however, model calculations predict a temporary drop in the rate of ascent up the giant branch when the burning shell reaches the discontinuity. They therefore predict a peak in the distribution of number versus magnitude along the giant branch in a real cluster.

One puzzling feature of giant branches in globular clusters is an apparent gap that occurs at a luminosity below the mean luminosity of the horizontal branch (Sandage et al 1968). If a large gap occurred along the giant branch of only a few clusters or if such a gap occurred in all clusters but at a random location in any one cluster, then one might ascribe the gap to a statistical fluctuation in stellar birthdates and/or in the initial distribution in number versus mass. In fact, there are many smaller "gaps" and "humps" that are undoubtedly due to such fluctuations (Iben 1968b). However, the gap just below the luminosity level of the horizontal branch seems to be a statistically significant characteristic of many clusters and may therefore be an indication that the evolution of a real star speeds up dramatically over a finite luminosity interval and then slows down again, in contrast to the predictions of standard giant models whose rate of climb increases smoothly and monotonically with increasing luminosity. An attempt by Demarque et al (1972) to account for a gap relies on rapid rotation in the core.

3.2 *Consequences of the Helium Flash*

Computations carried through the helium flash stage (Schwarzschild & Härm 1962, 1964, 1966, Thomas 1967, Hartwick et al 1968, Demarque & Mengel 1972) have demonstrated fairly conclusively that the procedure for constructing horizontal branch models adopted by a large number of investigators (e.g., Nishida 1960, Hayashi et al 1962, Suda & Virgopia 1966, Faulkner 1966, Rood 1970, Castellani et al 1969, 1971) is an adequate one. The detailed computations suggest that mixing between matter containing products of helium burning and matter beyond the edge of the hydrogen-exhausted core (as discussed by Sugimoto 1964 and Edwards 1970) does not take place. They show further that the structure of a model star, after electron degeneracy has been lifted and evolution again proceeds on a nuclear-burning time scale, may be reasonably well approximated by a static model with an initial helium core equal in mass to that of the helium core just prior to the helium flash. The error introduced by neglecting the production of carbon during the flash stage is not significant. On the chance that, despite the theoretical estimates, mixing does occur during the helium flash, Rood (1971) and Peterson (1972) have investigated the effect of such mixing on horizontal branch models.

3.3 *The Zero Age Horizontal Branch (ZAHB)*

The initial location of a model on the horizontal branch depends on at least five parameters: the mass in the helium core; the total mass of the star; the abundance of CNO elements in the hydrogen-burning shell; the opacity sources in the hydrogen-rich envelope, and the helium to hydrogen ratio in the envelope. If it may be assumed (as in all calculations to date) that the abundance of CNO elements is monotonically correlated with the abundance of opacity sources, then the number of parameters reduces to four. If it is assumed that currently adopted neutrino loss rates are correct and that core mass can be specified as a unique function of the envelope helium abundance and of the abundance of CNO elements, then the parameter set reduces to three: total stellar mass, envelope helium abundance, and initial abundance of CNO elements.

All other things being equal, lower Z (lower abundance of CNO elements and/or lower envelope opacity) means a bluer initial location on the horizontal branch (Faulkner 1966). For sufficiently large Z, subsequent evolution is completely confined to the giant branch (Faulkner & Cannon 1973). This result is in accord with the distributions of stars along the giant branches of population I clusters (Cannon 1970), which show a "clump" of stars along the giant branch at a luminosity level consistent with theory. Not until Z is decreased into the population II range do initial models of reasonable mass appear in the vicinity of the horizontal branches defined by stars in most globular clusters (Faulkner 1966).

The most complete studies of ZAHB morphology using reasonably up-to-date physics are those of Rood (1970) and Gross (1971). These studies show that, for fixed Y and fixed surface temperature T_e, the smaller Z is, within the population II range (say 10^{-3} to 10^{-5}), the more luminous is the initial horizontal branch model. For fixed Z and fixed T_e, larger Y means a larger luminosity. For fixed Y and Z, decreasing M moves an initial model from the red to the blue and from higher to lower luminosity. The limit, of course, occurs when $M = M_{\text{core}}$ and the limiting model lies on the main sequence for pure helium stars, far to the blue of the hydrogen-burning main sequence (e.g., Deinzer & Salpeter 1964, Devine 1965, Caloi 1972).

3.4 *Horizontal Branch Evolution*

Evolution along the horizontal branch during core helium burning has received extensive attention (e.g., Nishida 1960, Hayashi et al 1962, Nishida & Sugimoto 1962, Osaki 1963, Iben & Faulkner 1965, 1968, Faulkner & Iben 1966, Rood & Iben 1968, Hartwick et al 1968, Iben & Rood 1970, Castellani et al 1969, 1971, Demarque & Mengel 1972, 1973, Sweigert & Demarque 1972, Sweigert et al 1973, Sweigert & Gross 1973).

The primary direction of evolution in the H-R diagram has been found to depend strongly on the chosen helium abundance in the hydrogen-rich envelope (Iben & Faulkner 1965, 1968, Faulkner & Iben 1966). It depends also on the choice of the heavy element abundance parameter and on the mass in the hydrogen-rich envelope relative to the mass in the hydrogen-exhausted core.

As a general rule, the larger the contribution L_{H} of the hydrogen-burning shell to the luminosity relative to the contribution L_{He} of the helium-burning core, the greater is the tendency for the direction of evolution to be from red to blue. Consider, for example, the effect of increasing Y for fixed total mass and fixed Z. When Y increases, the mass of the initial hydrogen-exhausted core becomes smaller and L_{He} consequently decreases. At the same time, the number of particles in the envelope decreases. The envelope therefore contracts, temperatures rise at the base of the hydrogen-burning shell and L_{H} increases. Thus, as Y increases from zero past some critical value Y_{c}, the predominant direction of evolution switches from going from blue to red to going from red to blue. The value of Y_c depends on Z and on the range of surface temperatures considered. For $Z = 10^{-4} \rightarrow 10^{-3}$, Y_c is of the order of 0.2–0.25.

In a similar fashion, if both Y and Z are fixed, L_{He} remains fixed but L_H decreases as total mass is decreased and temperatures at the base of the hydrogen-burning shell drop. Hence, the smaller the mass in the hydrogen-rich envelope and therefore the bluer the initial location on the horizontal branch, the greater is the tendency for evolution to be primarily from blue to red. The dependence on total mass is shown in Figure 1 for particular choices of composition parameters and initial core mass (Strom et al 1970). Model stars with masses somewhat lower than those shown in Figure 1 evolve in a manner similar to that of pure helium models; that is, nearly vertically in the H-R diagram (Demarque & Mengel 1972).

A major uncertainty in horizontal branch evolution is the extent of the mixing region in the hydrogen-exhausted core. Overshoot and semiconvection appear to be

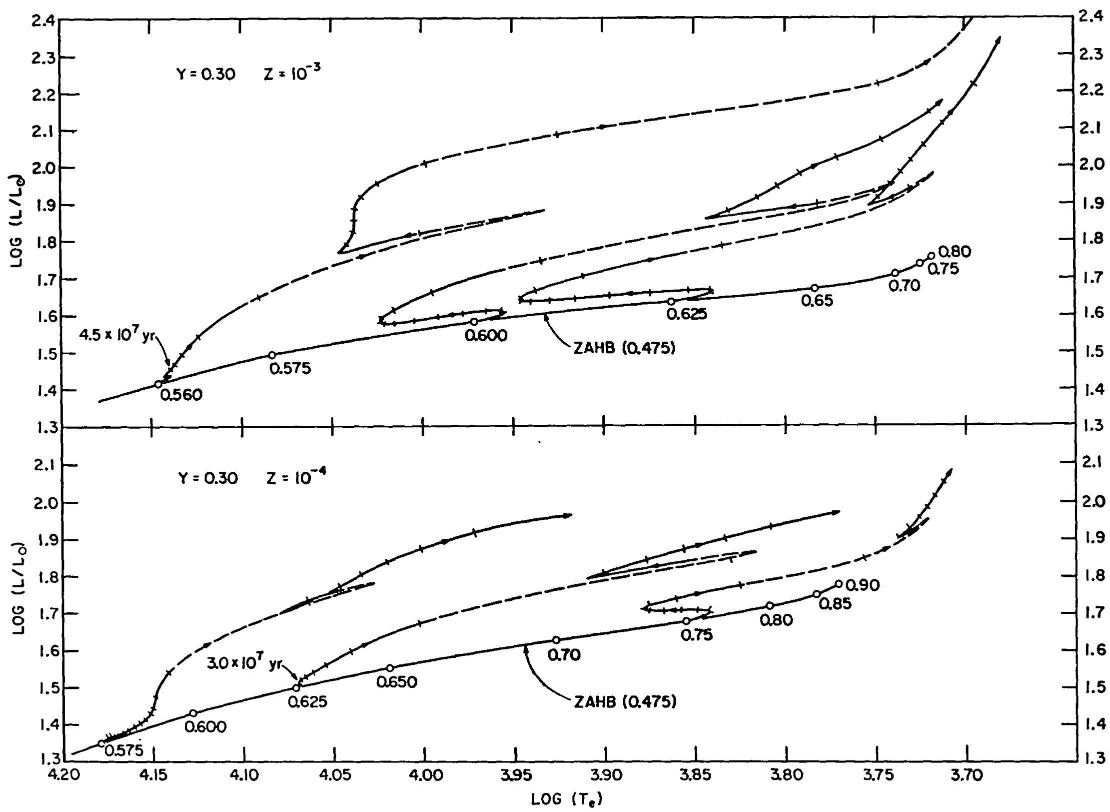

Figure 1 Horizontal branch, suprahorizontal branch, and asymptotic branch evolutionary tracks (Strom et al 1970). As each model moves off of the ZAHB along a solid track, helium burns at the center and hydrogen in a shell. The second solid portion of each track describes the path as helium burns in a thick shell and hydrogen continues to burn in a thin shell. Tick marks along each track are placed at intervals of 10^7 yr. Motion along the dashed portion of each track is rapid relative to motion along the solid portions. Model mass in solar units is indicated beside each track.

of significant importance (e.g., Castellani et al 1971, Schwarzschild 1970, Demarque & Mengel 1972, Sweigert & Demarque 1972, Sweigert & Gross 1973). During the early stages of horizontal branch evolution, convective overshoot, when treated in the manner of these authors, increases the size of the region in which full mixing occurs by over 50% in relation to the size of the region obtained when overshoot is neglected. Then a semi-convective region develops and continues to maintain partial mixing over a region comparable in mass to the mass of the fully mixed convective core. The net result is that, during most of the horizontal branch phase, mixing occurs over a region of mass roughly double the mass of the convective region obtained when semiconvection is neglected. The major consequences of observational significance are that, for some choices of composition ($Y \sim 0.3$, $Z \sim 10^{-3}$), the track length in the H-R diagram and the lifetime during the core helium-burning phase are both essentially doubled.

To a first approximation, horizontal-branch lifetime t_{HB} is a function only of the initial mass M_{c} of the helium core and is given by

$$\log t_{\mathrm{HB}} \approx 0.74 - 2.2(M_{\mathrm{c}} - 0.5) + \log f \tag{4}$$

where t_{HB} is in units of 10^7 yr and M_{c} is in solar units. When convective overshoot and semiconvection are neglected (Iben & Rood 1970), $f = 1$. When these effects are included in the current fashion, $f \approx 2$. Inserting the approximation for M_{c} from equation (2) into expression (4) gives

$$\begin{aligned}\log t_{\mathrm{HB}} \approx\ & 0.795 + 0.506(Y - 0.3) \\ & + 0.022(\log Z + 3) + 0.077(M - 0.8) \\ & + \log f\end{aligned} \tag{5}$$

Even when semiconvection and overshoot are included, evolutionary tracks extend over an interval in $\log T_e$ that is short compared to the widths of several horizontal branches observed in nature. This fact suggests that it may be necessary to assume a spread in mass among stars along the horizontal branch in order to account for the observed spread in color. A comprehensive discussion of the need for a mass spread is given by Rood (1973) who has also, for the first time, treated the theoretical data in such a way that it may be directly compared with observed distributions. Rood constructs synthetic horizontal branches by assuming a nearly Gaussian spread in masses and by plotting the position of each model at a randomly chosen time in its horizontal branch life.

Further evidence for a mass spread is discussed by Newell (1973), who also introduces an observational feature that no theoretical treatment has thus far been able to explain. This new feature is an apparent gap in the distribution in number versus T_e along the horizontal branch that occurs at $\log T_e \approx 4.11$ and another, weaker gap that occurs at $\log T_e \approx 4.33$. The demonstration of the existence of a distinct gap in the cluster NGC 6752 (Cannon 1973) corroborates the evidence adduced by Newell (1973) from an examination of field horizontal branch stars.

Since track morphology varies continuously with M (for given Y, Z, and M_{C}), one cannot rely on an abrupt change in track morphology when M is varied past some critical value in order to account for the gaps. It may be that stars to the blue of

$\log T_e \approx 4.1$ are not core helium-burning stars, but are stars in a double-shell stage undergoing thermal flashes in the interior. Or the gap may be due to a real discontinuity in the conversion from $B-V$ to T_e that has not been properly taken into account; with a proper conversion, the gap might possibly disappear.

To permit zeroth order comparisons with the observations, and to foster a feeling for the composition dependence, it is useful to present rough analytic approximations to the theoretical results. Of particular use are the luminosity L_{RR} and the mass M_{RR} of a horizontal branch model which, at a point midway through its career as a core helium burner, reaches some specific temperature T_e within the RR Lyrae instability strip. If the temperature is specified by $\log T_e = 3.85$, then one has (data from Iben & Rood 1970, as approximated by Iben 1971)

$$\log L_{\mathrm{RR}} \approx 1.75+2.16(Y-0.3)-0.05(\log Z+3)+3.1(M_{\mathrm{c}}-0.5) \tag{6}$$

$$\log M_{\mathrm{RR}} \approx -0.165+0.34(Y-0.3)-0.076(\log Z+3)+1.16(M_{\mathrm{c}}-0.5) \tag{7}$$

On inserting M_{c} from (2), one obtains

$$\log L_{\mathrm{RR}} \approx 1.67+1.45(Y-0.3)-0.081(\log Z+3)+0.11(0.8-M_{\mathrm{RR}}) \tag{8}$$

$$\log M_{\mathrm{RR}} \approx -0.194+0.073(Y-0.3)-0.088(\log Z+3)+0.04(0.8-M_{\mathrm{RR}}) \tag{9}$$

These expressions are extremely rough and should be used only for orientation purposes.

3.5 *Suprahorizontal Branch and Asymptotic Branch Evolution*

As helium becomes exhausted at the center, a model star evolves on a thermal time scale along a contorted path in the H-R diagram above the horizontal branch proper. Once helium burning is established in a thick shell, a model then evolves from blue to red on a nuclear-burning time scale. During this phase, hydrogen burning in a thin shell continues to contribute to the luminosity (Iben & Rood 1970). As time progresses and the helium shell narrows, the rate of helium burning increases, and the hydrogen-burning shell goes out.

Where the star spends most of its time in the H-R diagram during the thick-shell phase is a function of total mass (see Figure 1). The lower the total mass, the bluer is the mean location. In all cases in which the mean location occurs far to the blue of the giant branch, evolution occurs far enough above the mean location of a core helium-burning model that one may legitimately speak of a "suprahorizontal branch" phase. The height (in log L) above the horizontal branch and the thickness (also in log L) of the suprahorizontal branch depends sensitively on the composition parameters (Strom et al 1970, Iben & Rood 1970). The existence in globular clusters of a suprahorizontal branch both in fact and in theory was first demonstrated by Strom et al (1970).

Except for the very lightest stars (envelope mass $\ll M_{\mathrm{star}}$), which evolve nearly vertically in the H-R diagram during the core helium-burning phase and then evolve quickly thereafter into the white dwarf phase (Demarque & Mengel 1972), all stars eventually evolve over to a position close to the "first giant branch" (defined by pure hydrogen-burning stars) and subsequently evolve upward in the H-R diagram, approaching ever more closely to the first giant branch. During the latter portion of the "asymptotic branch" phase, the hydrogen shell reignites and the helium- and hydrogen-burning shells move outward together, separated by a very small mass of nearly pure helium ($\Delta M \gtrsim 0.003 M_{\odot}$).

3.6 *Thermal Relaxation Oscillations On the Asymptotic Branch*

Several different types of instability occur as a light star climbs upward along the asymptotic branch, burning hydrogen in one shell and helium in another. The first of these is a "thermal" instability that has its origin in the nuclear-burning region of the star. The instability was first encountered by Schwarzschild and Härm (1965), and the hydrostatic relaxation oscillations that are a consequence of this instability have since been studied by several investigators (e.g., Rose 1966, 1967a,b, Weigert 1966, Schwarzschild and Härm 1967, Hoshi 1968, Rose & Smith 1970, Vila 1970, Faulkner & Wood 1972, Wood & Faulkner 1973, Sweigert 1971, 1973, Sweigert et al 1973).

The relaxation oscillations consist of very short periods of rapid variation in internal characteristics (one or more sharp, staccato "pulses") followed by very long

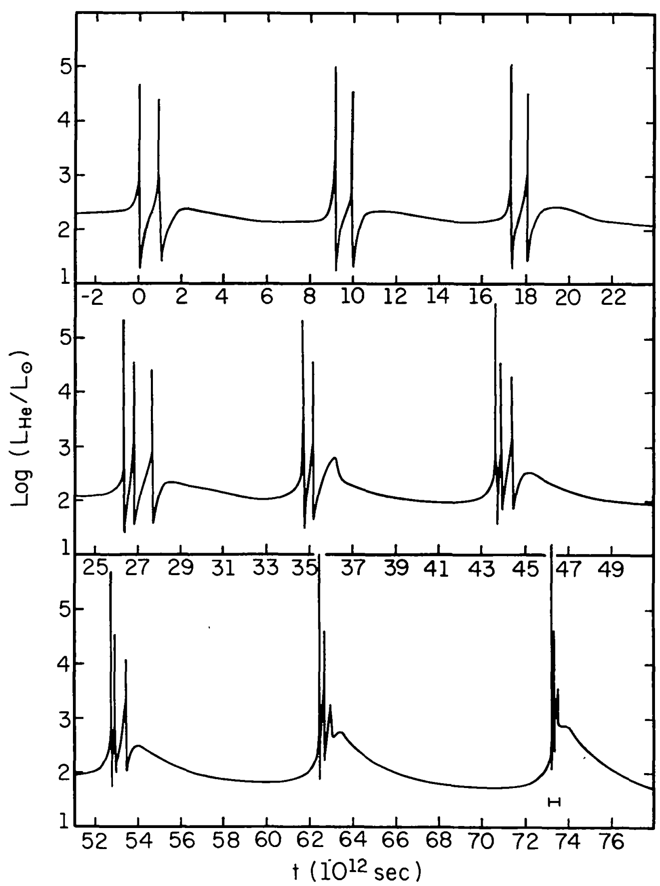

Figure 2 Power output from helium burning in a double shell source star of low mass that is undergoing relaxation oscillations as a consequence of a thermal instability (Schwarzschild & Härm 1967).

intervals of quiescent evolution. The finer detail depends on the mass of the star, on the mass of the carbon-oxygen core below the two burning shells, and possibly also on the composition. Some of this finer detail for a low-mass star of population II composition is shown in Figure 2 (from Schwarzschild & Härm 1967). One may note that: 1. each thermal pulse consists of a set of sub-pulses; 2. the maximum rate of energy production by the helium-burning shell during the most violent sub-pulse increases with each relaxation cycle; 3. the separation in time of sub-pulses decreases with each cycle; and 4. the time between major pulses increases with each cycle, the average time between major pulses being about 3×10^5 yr.

With each thermal runaway, a convective zone is generated by the large fluxes produced by helium burning. At maximum size this zone extends from a point near the base of the helium-burning shell to a point just below the hydrogen-helium discontinuity. In the work of Schwarzschild and Härm (1967), the outer edge of the convective zone actually reaches into the hydrogen-rich layer. More recent work (Sweigert 1973) suggests that this may not always be the case. However, in every instance so far studied, the *formal* convective zone always reaches extremely close to the hydrogen-helium discontinuity, and one might anticipate convective overshoot of some sort to possibly bring about mixing of helium-burning products into the hydrogen-rich layers and/or partial mixing of hydrogen down into the helium-carbon-oxygen zone.

The consequences of such mixing, though not yet demonstrated in realistic stellar models, have received considerable attention (Sanders 1967, Ulrich & Scalo 1972, Scalo & Ulrich 1973, Smith et al 1973). The exciting feature of these studies is that, via the sequence $C^{12}(p,\gamma)N^{13}(\beta^+\nu)C^{13}(\alpha,n)O^{16}$, large fluxes of neutrons may be generated to act on already-present seed nuclei to form *s*-process elements.

It is tempting to suppose that the sudden appearance of such elements at the surface of the star FG Sagittae (Langer et al 1973) is a demonstration that the envisioned sequence of events is really taking place in nature. The rapid variation in surface temperature of FG Sagittae (e.g., Herbig & Boyarchuk 1968) may also be a demonstration of another phenomenon that is associated with relaxation oscillations in model stars of low mass.

When the mass in the envelope above the hydrogen-burning shell is sufficiently small, the thermal instability can have a pronounced effect on surface temperature as well as on surface luminosity (e.g., Schwarzschild and Härm 1970, Vila 1970, Paczynski 1971a, Sweigert et al 1973). Schwarzschild and Härm (1970) show that, during each relaxation oscillation, models that are evolving on the asymptotic giant branch may loop away from this branch and return to it along a path that crosses the classical instability strip where W Virginis stars are found. Looping from the asymptotic branch over to and beyond the blue edge of the instability strip will not occur until the mass outside of the hydrogen-burning shell decreases below a critical value relative to the total mass of the model. This critical mass appears to be sensitive to the input physics (Sweigert 1973) and possibly also to composition parameters (Wallerstein 1970). Suprahorizontal branch models with hydrogen envelopes too small to permit evolution over to the asymptotic branch (Sweigert et al 1973) and model cores of newly formed planetary nebulae with extremely small hygrogen envelopes (e.g., Paczynski 1971a, Wood & Faulkner 1973) also manifest gyrations far to the blue of the giant branch while undergoing relaxation oscillations.

3.7 *Radial Pulsations, Envelope Relaxation Oscillations, and Mass Loss on the Asymptotic Branch*

For any given composition, the mean luminosity of an asymptotic branch star is a monotonic function of the mass in the carbon-oxygen core. In the absence of any mass loss, the maximum luminosity attainable on the asymptotic branch will be limited either by the total initial mass of the star (if this mass is less than the Chandrasekhar limit, $\sim 1.4M_\odot$) or by the onset of carbon burning at the center (if the total mass is greater than about $1.4M_\odot$), whichever comes first. The magic mass of $1.4M_\odot$ enters simply because temperatures and densities in the core do not become large enough to ignite carbon until the core mass approaches the limiting mass for a stable white dwarf.

The observations suggest that stars as cool as asymptotic branch stars become (beyond the red giant tip proper) will lose mass at a very high rate. There are also theoretical indications that such stars will lose mass. For example, comparisons between models and horizontal branch and red giant branch stars in globular clusters suggest that red giants lose mass to the tune of about $0.2M_\odot$ in 10^8 yr (Iben & Rood 1970) and it might be expected that cooler and more luminous asymptotic branch stars would lose mass at an even greater rate via the same ejection mechanism, whatever that mechanism might be. For sufficiently cool stars, carbon grains are expected to form at the surface and be driven out away from the star by absorbing momentum from the radiation field (e.g., Wickramasinghe et al 1966, Donn et al 1968, Gehrz & Woolf 1971).

Roxburgh (1967), Lucy (1967), Paczynski & Ziolkowski (1968) have pointed out that the ionization energy of the matter in a very extended giant envelope is greater than the binding energy between this envelope and the condensed core of the star. Thus, if some instability could be found to initiate mass outflow, there would be enough stored energy to drive this outflow, if this store could be tapped.

Detailed dynamic calculations, however, show that the behavior of giant envelopes is sufficiently non-adiabatic that the ionization-energy reservoir is not tapped directly (Keeley 1970, Smith & Rose 1972, Wood 1973a,b). That mass loss which occurs prior to the ejection of a planetary nebula occurs as a consequence of shocks generated during radial pulsations that are driven by that portion of the hydrogen ionization zone that lies near the outer edge of the extended convective region in the envelope (Keeley 1970, Wood 1973a,b).

Both Keeley and Wood show that, as luminosity is increased in the range $\log L \approx 3.4-4.1$, pulsation in overtone modes gives way to pulsation in the fundamental mode. Shocks are formed at an interface between an infalling radiative region and an outward moving convective region, and some mass loss is a consequence, although it is not clear precisely how much. The switch from overtone to fundamental as the favored mode is correlated with the fact that, as luminosity and core mass increase, the mass above the convective envelope (where much of the driving occurs) also increases, so that more of the volume of the star is involved in the pulsation.

As the mean luminosity continues to increase, regular pulsation gives way to an envelope relaxation oscillation (Smith & Rose 1972, Wood 1973a). In each cycle, a distinct outward moving shell forms and attains velocities approaching the velocity of escape from the rest of the star. The nearly detached shell shocks the material ahead of it, causing true mass loss. Afterwards the shell cools and collapses back onto the main body of the star, and then rebounds and moves outward again to start a new cycle.

Wood (1973b) finds that there is a critical luminosity beyond which the velocities in the outward moving shell exceed escape velocities and a planetary nebula containing essentially all of the envelope (except for about $0.0003M_{\odot}$ that remains just outside of the hydrogen burning shell) is formed. It is interesting that the ejection of the envelope occurs at a luminosity below the critical luminosity for radiation driven mass flow, $L = 4\pi cGM/\kappa$. This is comforting since treatment of the radiation-driven outflow cannot be very definitive when much of the matter is in a convective region with a degree of super-adiabaticity that is not, by a wide margin, accurately calculable (Paczynski & Ziolkowski 1968, Paczynski 1969, Faulkner 1970, Finzi & Wolf 1971, Sparks & Kutter 1972, Cassinelli & Castor 1972, Zytkow 1972, Joss et al 1973).

Model studies of the core left behind after the formation of a planetary nebula are plentiful. Since the review by Salpeter (1971), a number of investigations have appeared (Paczynski 1971a, Smith & Rose 1972, Sparks & Kutter 1972, Faulkner & Wood 1972, Wood & Faulkner 1973, Joss et al 1973). These studies show that, after a brief period of burning in both hydrogen and helium shells, during which the thermal instability causes relaxation oscillations, a model star descends into the region of white dwarfs.

3.8 *Pulsation Theory and RR Lyrae Stars*

Comparisons between results of theoretical radial pulsation calculations and the observations provide estimates of properties of low-mass stars that complement those given by evolutionary calculations. For discussions of theoretical light curves versus observed light curves see Christy (1966a,b). For comparisons between results of linear pulsation calculations and the observations see van Albada and Baker (1971, 1973), Iben (1971), Fernie (1972), and Ledoux (1974).

Perhaps the most basic of all relationships established by theoretical pulsation calculations is that between period P, luminosity L, surface temperature T_e, and stellar mass. This relationship is nearly independent of composition parameters and is approximately (Iben 1971)

$$\log P_F \approx -0.340+0.825(\log L-1.7)-3.34(\log T_e-3.85)-0.63(\log M+0.19) \tag{10}$$

where L and M are in solar units and P_F is the period of the fundamental mode in days. The period of the first overtone or "first harmonic" P_H is related to P_F by $\log P_H \sim \log P_F - 0.127$.

Given P_F and T_e, one may use (10) to determine a relationship between M and L. For example, in the cluster M3 the transition between Bailey c-type variables (first overtone pulsators) and Bailey ab-type variables (fundamental pulsators) occurs at $\log P_F \approx -0.31$ and $\log T_e \approx 3.83$. (See Dickens 1971 and Iben & Huchra 1971 for the ingredients of this estimate.) Hence, we have

$$\log M_{tr} \approx -0.13 + 1.31(\log L_{tr} - 1.7) \tag{11}$$

If $\log L_{tr} = 1.7$, then we have $M_{tr} \approx 0.74 M_\odot$; if $\log L_{tr} = 1.6$, then $M_{tr} \approx 0.55 M_\odot$. Both sets of masses and luminosities are roughly consistent with estimates suggested by evolution theory.

Additional information may be obtained by comparing the properties of the bluest variables in a cluster with the properties of theoretical blue edges for pulsation in the first harmonic mode. The relationship between period P_{HBE}, T_e, and the composition parameter Y is nearly independent of M and of Z and is approximately (Tuggle & Iben 1972)

$$\begin{aligned} Y \approx\ & 0.22 + 5(\log T_e - 3.863) \\ & + (\log P_{\mathrm{HBE}} + 0.55)/(1.5 + 5Y) \end{aligned} \tag{12}$$

With $\log P_{\mathrm{HBE}} \approx -0.55$ and $\log T_e \approx 3.863$ (Dickens 1971), $Y \approx 0.22$ for stars in M3.

Another relationship which holds along a first harmonic blue edge is (Tuggle & Iben 1972)

$$\begin{aligned} \log L_{\mathrm{HBE}} \approx\ & 1.62 + 0.9(\log P_{\mathrm{HBE}} + 0.55) \\ & + 0.7 \log(M/0.6) + 0.88(Y - 0.2) \end{aligned} \tag{13}$$

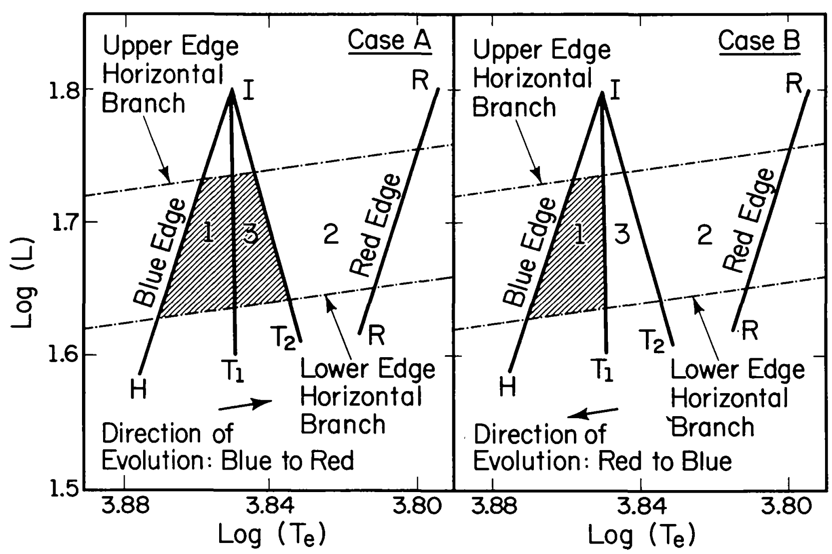

Figure 3 Schematic showing the upper and lower boundaries of the horizontal branch and the red and blue boundaries of the instability strip for radial pulsations. In region 1, the favored pulsation mode is the first harmonic; in region 2, it is the fundamental. In region 3 pulsation continues in whichever mode is initiated.

With $Y \approx 0.22$, $\log P_{\rm HBE} \approx -0.55$, and $M = 0.65 M_\odot$, one has $\log L_{\rm HBE} \approx 1.66$, again more or less consistent with results of stellar evolution theory. If one supposes that L_{tr} in (11) is nearly the same as $L_{\rm HBE}$ in (13), then $M \approx 0.55 M_\odot$ and $L_{tr} \approx L_{\rm HBE} \approx 1.60$.

Since blue edge relationships depend sensitively on opacity (compare Tuggle & Iben 1972 with Iben 1971), the results presented here cannot be considered definitive.

Another derivative of pulsation theory has to do with a "transition" region which, for any choice of mass and composition, extends downward in the H-R diagram from the intersection between the fundamental and first harmonic blue edges, as shown schematically in Figure 3 (region 3 bounded by the open angle T1-I-T2). This region is defined by Christy (1966a,b) as one within which the character of the pulsation that is achieved after many cycles of calculation depends on the initial conditions. If motion is begun in the pure fundamental mode, final motion remains in the fundamental mode; if motion is begun in the first harmonic mode, final motion is also in the first harmonic mode. Thus pulsation in the transition region is characterized by "hysteresis."

In the area between the transition region and the first harmonic blue edge, final motion is always in the first harmonic mode (area 1 bounded by angle H-I-T1). In the area to the red of the transition region (area 2 bounded by I-T2 and RR), final motion is in the fundamental mode.

The characteristics of the transition region are, as yet, incompletely explored. Brute force calculations (limited to following motion for only a few hundred cycles) seems to define a region considerably larger than is defined by a more sophisticated technique that permits one to search for full amplitude periodic solutions and to test these full-amplitude solutions for stability (Baker and von Sengbush 1969; von Sengbush 1973, Stellingwerf 1974). It is possible that, for some compositions, the transition region has (as far as observational consequences are concerned) zero measure. That is, the transition *region* may, for all practical purposes, reduce to a transition *edge*.

On the other hand, van Albada and Baker (1973) demonstrate that, if the region of hysteresis is of finite width, one may account in a very natural way for the two Oosterhoff groups into which clusters fall in terms of a difference in the direction of evolution along the horizontal branch. Their argument may be very simply summarized. Let us assume, as in the left hand panel of figure 3, that all stars evolve from blue to red (case A). Any star passing from region 1 into region 3 will continue to pulsate in the first harmonic mode, not switching into the fundamental mode until it passes into region 2. In a similar fashion, if all stars evolve from red to blue (case B), then a star evolving from region 2 into region 3 will continue to pulsate in the fundamental mode until it reaches region 1, whereupon it will switch into the first harmonic mode. Thus, if all things but the direction of evolution are the same in the two hypothetical cases, the number ratio of c-type variables to ab-type variables will be larger in case A than in case B and the average fundamental period will also be larger in case A than in case B.

These differences mimic those between clusters ordered according to the Oosterhoff

(1939) classification scheme. Van Albada and Baker take the preceding argument as evidence that, in Oosterhoff type I clusters, horizontal branch stars evolve from red to blue, and in Oosterhoff type II clusters, they evolve from blue to red.

It is interesting that these inferences about direction of evolution agree with the results of evolution calculations as applied to the interpretation of observational characteristics of two prototype clusters, M3 and ω Centauri. Iben and Rood (1970) argue that in M3 (Oosterhoff type I) evolution proceeds in both directions, but predominantly from red to blue; in ω Cen (Oosterhoff type II), because of low Z, the bulk of horizontal branch stars evolve far to the blue of the pulsational instability strip, but eventually all pass through the instability strip from blue to red during a transitional phase between the horizontal branch phase and the suprahorizontal branch or early asymptotic branch phase. The large overlap in color between c-type and ab-type variables in ω Cen is probably due to a large spread in the mass and luminosity of stars passing through the instability strip. It may also in part be due to a spread in composition parameters.

3.9 *The Age and Composition Problem for Globular Clusters*

From the standpoint of cosmology and from the standpoint of galactic dynamics and nucleosynthesis, the two most important characteristics of globular cluster stars are age and initial composition. It is probably safe to assume that the spread in birthdates among stars in a given cluster is small compared to the age of the cluster. However, it is not at all unlikely that the initial composition may vary appreciably from one star to another in a given cluster. What is lacking, of course, is a set of careful *spectroscopic* estimates of heavy element abundances at the surfaces of *several* stars in a given globular cluster.

When compared with results of theoretical giant branch calculations, the photometric evidence for one cluster, ω Cen, suggests that the spread in heavy elements may be quite large. At any given magnitude on the giant branch of ω Cen, the spread in color (Dickens & Wooley 1967) is far too large to be due either to errors in photometry (Cannon 1973) or to differences in mass or helium abundance (Rood 1972a). The only parameter remaining is the abundance of heavy elements (Castellani 1973).

It is easy to see how a spread in composition might come about. For example, one might assume that matter in galactic protoclusters was contaminated by heavy elements spewed out from the galactic plane following an early, active phase of star formation and evolution. The grains formed by these heavy elements in the protocluster would contribute to rapid cooling and thereby possibly trigger star formation throughout the cluster. Supernova explosions within the cluster would still further contaminate gaseous matter in the cluster. The net result would be an inhomogeneous distribution of heavy elements in the final stellar component that survived to the present.

Having admitted a variation in element abundances among stars in a given cluster, the best one can hope for on comparing theoretical models with the observations is very rough estimates of *mean* abundances. There are at least five ways of estimating the mean helium abundance.

1. The first procedure is based on (12) and yields $Y \approx 0.22$ for variable stars in the cluster M3. Only three other clusters (M5, M15, ω Cen) contain enough variables for this method to be applied with any confidence. It has been suggested that the intrinsic color at the blue edge of the instability strip is very much the same for all four clusters (Sandage 1970). Since the location of a blue edge in the T_e coordinate is fairly insensitive to Z, but highly sensitive to Y, this apparent constancy might be taken to indicate that the variation in mean Y among the four clusters is very small. However, since the conversion from $B-V$ to T_e may be very sensitive to Z, even when Z is in the population II range, a constancy in $B-V$ might in fact imply a large variation in Y. Among the weak points of the method are (*a*) the uncertainty in the conversion between $\langle B \rangle - \langle V \rangle$ and T_e for any choice of composition (see Böhm-Vitense & Szkody 1973) and (*b*) uncertainty as to the reddening correction needed to go from observed color to intrinsic color.

2. A second approach involves a comparison between theoretical luminosity functions (e.g., Simoda & Iben 1970, Hejlesen 1972) and observed luminosity functions (Simoda & Kimura 1968, Hartwick 1970, and Simoda & Tanikawa 1970, 1972). Again, the evidence (Simoda 1972) points to a high value for Y (> 0.2), but the precision is not excessive. Among the weak points in this method is the need to specify boundaries to cluster age.

3. A third procedure involves comparisons in the $\log g$–$\log T_e$ plane between properties of initial horizontal branch models and model atmosphere mediated interpretations of observational features of horizontal branch stars. Gross (1971) estimates $Y > 0.3$. The major weak point in this method is the uncertainty in the model atmosphere analysis.

4. A fourth method involves comparison between theoretical lifetimes of model stars on the giant branch and on the horizontal branch with observed number ratios of red giant stars and horizontal branch stars (Iben 1968b, Iben & Rood 1969). Setting L_{RG} in (3) equal to L_{RR} in (8), and approximating $M = M_{\mathrm{RR}} = 0.6$ in these equations, one has

$$\log t_{\mathrm{RG}} \approx 0.78 + 2.07(0.3 - Y) + 0.028(\log Z + 3) \tag{14}$$

as an estimate of the time required by a star to evolve up to the red giant tip from a point roughly equal in luminosity to the luminosity of a blue RR Lyrae star ($\log T_e \approx 3.85$). Combining with (5) one has

$$\log(t_{\mathrm{HB}}/t_{\mathrm{RG}}) \approx 2.58(Y - 0.3) - 0.0006(\log Z + 3) + \log f \tag{15}$$

as a measure of the ratio of horizontal branch lifetime to the lifetime of a red giant above a specified luminosity level ($L > L_{\mathrm{RG}} = L_{\mathrm{RR}}$). The ratio of lifetimes should be equal to the number ratio of horizontal branch stars to red giant stars more luminous than the bluest RR Lyrae stars (provided these variables are truly core helium burners). Setting $t_{\mathrm{HB}}/t_{\mathrm{RG}} = N_{\mathrm{HB}}/N_{\mathrm{RG}} = R$ and neglecting the very weak Z dependence, one has finally

$$Y \approx 0.3 - 0.39 \log(f/R) \tag{16}$$

Estimates for about a dozen clusters (Iben et al 1969) scatter about a mean of $R \approx 0.9$. If semiconvection and convective overshoot are neglected ($f = 1$), (16) suggests $Y \approx 0.28$. If the effect of including convective overshoot and semiconvection is to make $f \approx 2$, then (16) with $R = 0.9$ suggests $Y \approx 0.16$.

5. A final method of estimating Y is to combine results of pulsation and evolution theory. One procedure is to replace L_{tr} and M_{tr} in (11) by estimates of these quantities given by evolution theory. Using the *extremely* rough approximations (Iben 1971) $d\log L/d\log T_e \approx 0.4(\log Z+2)$ and $d\log M/d\log T_e \approx -0.2+0.3(\log Z+3)$ to extrapolate to $\log T_e = 3.83$ and setting $(0.8-M_{\mathrm{RR}}) = 0.2$, (8) and (9) yield

$$\log L_{tr} \approx 1.70-1.45(0.3-Y)-0.089(\log Z+3) \tag{17}$$

$$\log M_{tr} \approx -0.182-0.073(0.3-Y)-0.094(\log Z+3) \tag{18}$$

When these are inserted into (11), one obtains

$$Y \approx 0.27+0.012(\log Z+3) \tag{19}$$

This equation is restricted to Oosterhoff type I clusters, where the second term should be small. More accurately, it applies strictly only to the cluster M3 where there is a well defined transition edge at $\log P_{tr} \approx -0.31$ and $\log T_e \approx 3.83$.

Still another combination establishes an estimate of a lower limit on Y. By insisting that M_{tr} given by (18) be larger than the core mass M_c given by (2a) one obtains a lower limit on L_{tr}:

$$\begin{aligned}\log L_{tr} > {} & 1.63+0.158(0.3-Y) \\ & -0.007(\log Z+3)\end{aligned} \tag{20}$$

Then, combining (20) with (18), one has

$$Y > 0.256+0.051(\log Z+3) \tag{21}$$

an expression again designed with the cluster M3 primarily in mind.

One may conclude that the initial helium abundance of globular cluster stars is distinctly larger than zero and is possibly not inconsistent with the value $Y \approx 0.23$ given by the simplest versions of big bang element synthesis (Peebles 1966; Wagoner, Fowler & Hoyle 1967). The high value of Y given by method 3 possibly suggests deficiencies in the model atmosphere interpretation of observed spectral distributions. The low value of Y given by method 4 when convective overshoot and semiconvection are included in a specific approximation might possibly indicate that this approximation overemphasizes the effects of convective overshoot and semiconvection.

An estimate of cluster age is tied to an estimate of the mean value of Y for stars in the cluster. One must first estimate the abundance of heavy elements in a typical cluster star and then, via one or more of the methods just outlined, estimate a mean Y and, simultaneously, estimate horizontal branch luminosity. If the location of the cluster turnoff point is known relative to the location of the horizontal branch, one has finally an estimate of the absolute luminosity at cluster turnoff. This absolute luminosity, in conjunction with an estimated Y and Z, permits an estimate of cluster age.

The turnoff point in a cluster such as M3 occurs at $\log T_e \approx 3.81$ and at a luminosity $L_{\rm to}$ such that $\log L_{\rm to} \approx \log L_{3.81} - 1.36$, where $L_{3.81}$ is the luminosity of a typical horizontal branch star of the same color as stars at turnoff. Using $d\log L/d\log T_e \approx 0.4(\log Z+2)$, and extrapolating (17), one obtains

$$\log L_{\rm to} \approx 0.35 - 1.45(0.3 - Y) - 0.096(\log Z + 3) \tag{22}$$

Inserting this in (1) gives

$$\log t_{\rm c} \approx 0.035 + 2.085(0.3 - Y) - 0.034(\log Z + 3) \tag{23}$$

With $Y \approx 0.2$ and $Z \approx 10^{-3}$, this expression yields $t_{\rm c} \approx 17.4$ billion years; With $Y \approx 0.3$ and $Z \approx 10^{-3}$, $t_{\rm c} \approx 10.8$ billion years.

If one chooses to accept the lower limit on L_{tr} given by (20), and agrees that $\log L_{\rm to} > \log L_{tr} - 1.36$, then

$$\log t_{\rm c} < 0.12 + 0.42(0.3 - Y) - 0.15(\log Z + 3) \tag{24}$$

Finally, accepting the lower limit on Y given by (21),

$$\log t_{\rm c} < 0.122 - 0.17(\log Z + 3) \tag{25}$$

When $Z = 10^{-3}$, this expression yields $t_{\rm c} < 13$ billion years.

The methods just described cannot be used for clusters that do not have stars along the horizontal branch at colors similar to those at cluster turnoff. In clusters such as M13, the horizontal branch stars are too blue; in clusters such as 47 Tuc they are too red. The most populous cluster ω Cen must be treated with particular caution, for in this cluster stars on the "horizontal branch" at colors equivalent to colors of turnoff point stars may not be in the stage of core helium burning. In all of these cases, then, one must resort to another procedure.

An alternate way of estimating $L_{\rm to}$ is main sequence fitting (e.g., Sandage 1970, Hartwick & Vanden Berg 1973). It is difficult, however, to obtain a fiducial standard and to make adjustments for the differences between the composition characteristics of the standard and those of the cluster whose age is to be estimated. Concerning the establishment of a fiducial standard on "purely observational grounds," it is sufficient to point out that the location of nearby subdwarfs as argued by Cayrel (1968) differs considerably from the location as argued by Strom et al (1967).

4 EVOLUTION OF STARS OF INTERMEDIATE MASS

The term intermediate mass will be applied in this section to all stars which, *in the absence of mass loss,* would ignite carbon or oxygen in an electron-degenerate core. For a typical population I composition, this corresponds to stars of initial mass in the range 1.4–10$M_\odot$.

4.1 *Evolution to The Onset of Helium Burning*

Following the main sequence phase, stars of intermediate mass evolve rapidly to the region of red giants and begin to climb upward in luminosity along the relevant

"Hayashi" boundary. In stars initially less massive than about $2.25M_\odot$, electrons in the hydrogen-exhausted core become highly degenerate (ε_F = electron Fermi energy $\gtrsim 10\,kT$) before helium ignition occurs. As the core contracts, most of the gravitational potential energy that is released locally (and does not flow out via electron conduction or escape directly in the form of neutrinos) is converted into the non-thermal kinetic energy of the electrons. This helps keep down the rate at which core temperatures rise with increasing density and thus helps delay the onset of helium burning. Cooling via electron conduction and plasma neutrinos also helps keep the mean temperature in the electron-degenerate core from rising as rapidly with density as would occur if the electrons were not degenerate. The delay in helium ignition permits a star less massive than $2.25M_\odot$ to evolve upward along the giant branch to luminosities ($\sim 10^3\ L_\odot$) much greater than those achieved by this star during the main sequence phase.

In the density-temperature plane, the paths followed by the centers of all stars less massive than about $2.25M_\odot$ approach a common path, as is shown in Figure 4 (from Iben 1973a). This is a consequence of the fact that the density and pressure distributions in the inner parts of the hydrogen-exhausted core and the rate of energy

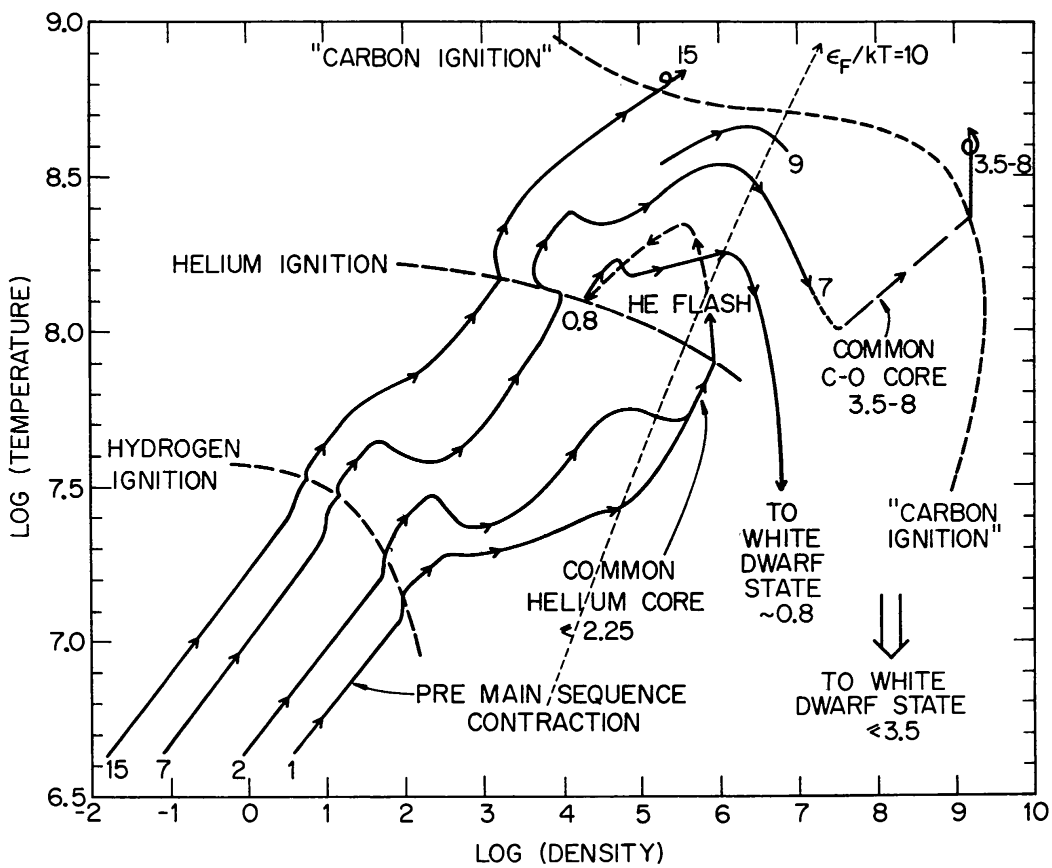

Figure 4 Tracks in the ρ-T plane traced out by the centers of stars of various masses (Iben 1973a).

production in the hydrogen-burning shell are approximately unique functions of the core mass. Since there must be a balance between energy input and energy outflow and since energy input (release of gravitational energy) is proportional to the rate at which core mass increases and is therefore almost solely determined by core mass, the temperature distribution in the core (which is determined by the rate of conductive and radiative outflow) is also almost solely determined as a function of core mass.

When core temperatures reach the ignition point for helium, a thermal runaway ensues and the star readjusts quickly to a new equilibrium state with a hydrogen-exhausted core in which electrons are no longer degenerate and with an envelope that is much less distended than at the red giant tip. In Figure 4, the path of central ρ and T during the helium flash is sketched for a low mass star (the dashed curve for $0.8M_\odot$ beginning and ending on the helium ignition curve). The paths of all intermediate-mass stars less massive than $2.25M_\odot$ are qualitatively very much the same during the flash, the initial and final positions on the helium-ignition curve being essentially independent of total stellar mass.

In stars more massive than $2.25M_\odot$, helium-burning temperatures are reached at the center before electrons become degenerate there. During the ensuing core helium-burning phase, hydrogen continues to burn in a shell at about the same rate as it did in the core during the main sequence phase. The rate at which helium is burned in the core determines the rate at which structural changes occur. Typically, the lifetime in the core helium-burning stage is 10–20% of the main sequence lifetime.

4.2 *Evolution in the H-R Diagram During Core Helium Burning*

The path of a typical model in the H-R diagram is shown in Figure 5 (from Iben 1972, 1974). Slow evolution during core helium burning takes place in two distinct regions: arcs *AB* and *CD* in Figure 5. The location of these two regions and the amount of time spent in each region are functions of mass and composition. For any choice of composition, as stellar mass is decreased, the location of the blue region moves toward the giant branch region, eventually merging with the giant branch. Thus, for a given composition, core helium burning occurs in two bands, one that roughly coincides with the locus of red-giant tips and another that breaks off from the giant-branch band at low luminosity and moves toward the blue with increasing luminosity, as illustrated in Figure 6 (Iben 1974).

The phase of rapid evolution on a thermal time scale that joints the two regions has been variously attributed to 1. the vanishing of convection over a large region of the envelope with a consequent readjustment to radiative equilibrium (Iben 1967); 2. a secular instability (Cox & Guili 1968; Lauterborn, Refsdal & Roth 1971); and 3. the occurrence of multiple solutions to the equations that describe static stellar envelopes that match onto helium cores (Lauterborn 1972).

Robertson (1971), Lauterborn, Refsdal & Weigert (1971), and Fricke & Strittmatter (1972) have shown that the extent to which evolution proceeds to the blue during the second major helium-burning phase (point *D* in Figure 5) is a very sensitive function of the hydrogen profile left behind by the receding convective core during the main sequence phase. For this reason, uncertainties as to behavior during the core helium-burning phase are aggravated by uncertainties in the extent and nature of mixing during the main sequence phase.

Another influence on the location of the blue limit is the choice of cross-section factor for the reaction $C^{12}(\alpha,\gamma)O^{16}$. A variation of this factor within experimentally allowable limits (Dyer 1973) leads to a variation of $\Delta \log T_e \sim 0.1$ in the location of the blue limit (Iben 1972). The larger the cross-section factor, the greater is the extent to which carbon is converted into oxygen and the longer is the duration of the core helium-burning phase. Thus, the greater the cross-section factor, the further the hydrogen-burning shell eats into the hydrogen profile and the further the star evolves toward the blue before rapid core contraction and envelope expansion set in and evolution proceeds back to the red (points D to E in Figure 5).

The extent to which evolution proceeds to the blue is influenced further by the degree of mixing in the core outside of the formal central convective region and by the rate of rotation. As in the case of less massive stars, overshoot and semi-

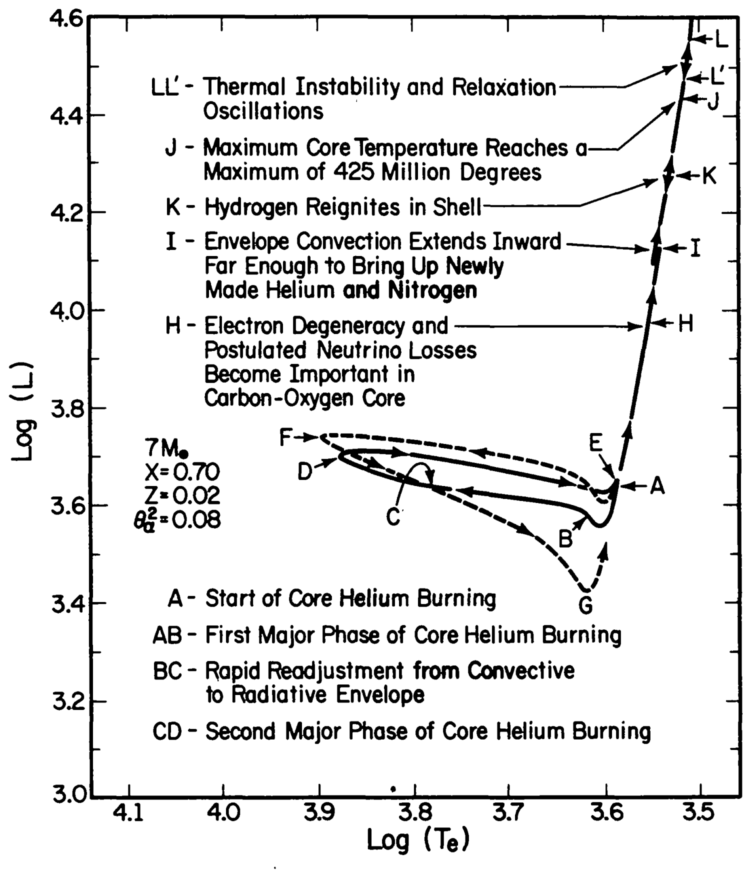

Figure 5 Path of a $7M_\odot$ model star in the H-R diagram during helium-burning phases (Iben 1974). The two major periods of core helium burning take place along the arcs AB and CD. Central helium vanishes along EF and hydrogen burning becomes unimportant along FG. The direction of evolution is reversed at point J and then reversed again at point K where hydrogen reignites. A thermal instability sets in and relaxation oscillations cause fluctuations in L as the mean value of L moves upward.

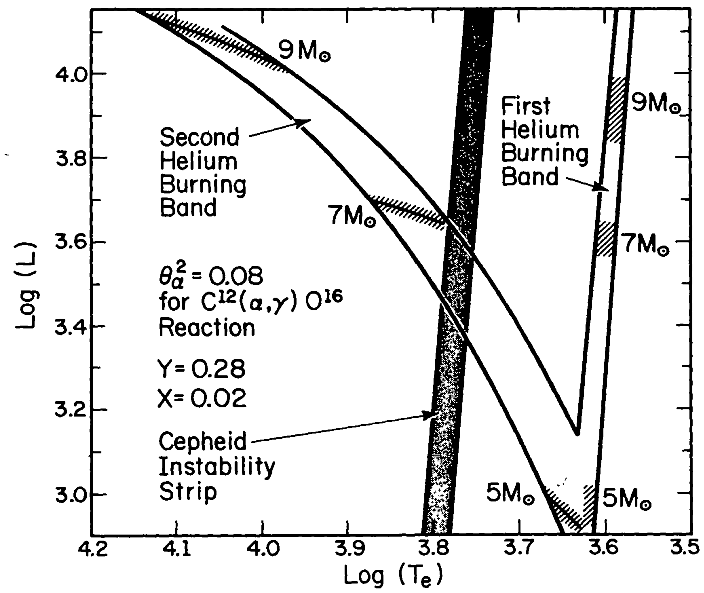

Figure 6 The two major helium-burning bands and a pulsational instability strip for $Y = 0.28$, $Z = 0.02$ (Iben 1974).

convection appear to play a role (e.g., Paczynski 1970, Robertson 1971). The enhancement of the fuel supply by an increase in the size of the effective mixing region and the consequent increase in the duration of the core helium-burning phase cause evolution to proceed further to the blue than is the case when these effects are not taken into account (Robertson 1971). How rotation affects evolution depends on which of many possible assumptions nature happens to choose. Both Kippenhahn et al (1970) and Meyer-Hofmeister (1972) find that, for a specific set of "plausible" assumptions, the effect of an altered initial hydrogen profile left behind after main-sequence burning is to increase the maximum extent to which an evolutionary track evolves to the blue.

Uncertainties in track morphology that are related to possible uncertainties in opacity have been studied by Robertson (1972), and by Fricke et al (1971, 1972). The effects of altering composition parameters have been explored by Schlesinger (1969), Robertson (1972), and Hallgren & Cox (1970), who show that, for a fixed stellar mass, a reduction in Y at constant Z leads to a reduction in the mean stellar luminosity during core helium burning, but to not much of a reduction in the surface temperature at which the blue limit occurs; a reduction in Z at fixed Y leads to an increase in mean luminosity and to an increase in the surface temperature of the blue limit.

There have not yet been sufficiently extensive and systematic studies to determine how all of the properties of the second helium-burning band in Figure 6 vary with composition parameters. However, the relationship between mass and luminosity within the band can be estimated as (Iben & Tuggle 1972)

$$\log L \approx 3.1 + 4(\log M - 0.7) - 4(X - 0.7) - 12(Z - 0.02) \tag{26}$$

where $0.6 \gtrsim X \gtrsim 0.8$, $0.01 \gtrsim Z \gtrsim 0.03$, and $3 < M/M_\odot < 10$. Equation (26) is an approximation to a more complex non-linear relationship and should be used primarily for orientation purposes.

The rapid changes in surface temperature that occur toward the end of the core helium-burning phase (dashed curve *EFG* in Figure 6) are not accompanied by large changes in luminosity. Hence, for a given composition, the dispersion in $\log L$ in the mass-luminosity relationship for core helium burners will be relatively small ($\Delta \log L \approx 0.1$ at constant M).

4.3 *Cepheids, Pulsation Theory, and the Question of Mass Loss*

Pulsation calculations yield a relationship between fundamental period P (in days), surface temperature T_e, stellar mass M, and luminosity L, which may be approximated as (Iben & Tuggle 1972)

$$\log P \approx 0.65 + 0.83(\log L - 3.25) - 0.63(\log M - 0.7) - 3.4(\log T_e - 3.77) \tag{27}$$

Using (27) or its equivalent and adopting estimates of T_e and L for thirteen Galactic Cepheids for which photometric estimates of absolute visual magnitude are possible and for which reddening corrections can be made (e.g., Sandage & Tammann 1969), pulsation masses M_{puls} can be derived (Cogan 1970; Rodgers 1970; Fricke et al 1971, 1972; Iben & Tuggle 1972). Choosing the mass-luminosity slope given by (26), the resulting mean relationship between M_{puls} and estimated L is

$$\log M_{\text{puls}} \approx 0.59 + 0.25(\log L - 3.25) - 5.4\,\delta \log T_e + 1.32\,\delta \log L \tag{28}$$

where $\delta \log T_e$ is the mean error committed in converting from observed color to T_e and $\delta \log L$ is the mean error committed in estimating $\log L$. That is, we have $(\log L)_{\text{true}} = (\log L)_{\text{estimated}} + \delta(\log L)$ and $(\log T_e)_{\text{true}} = (\log T_e)_{\text{estimated}} + \delta(\log T_e)$. Comparing (26) and (28) one obtains

$$\log(M_{\text{evo}}/M_{\text{puls}}) \approx 0.15 + (X - 0.7) + 3(Z - 0.02) + 5.4\,\delta \log T_e - 1.32\,\delta \log L \tag{29}$$

where M_{evo} is the value of M given by (26).

Arguments involving the position of blue edges in both the H-R diagram and in the P-L diagram suggest that the dependence on composition parameters and the uncertainty represented by $\delta \log T_e$ may be neglected relative to the possible error represented by $\delta \log L$ (Iben & Tuggle 1972). Hence, we have

$$\log(M_{\text{evo}}/M_{\text{puls}}) \approx 0.15 - 1.3\,\delta \log L \tag{30}$$

and one may infer that, unless there is a systematic error of order $\delta \log L \approx 0.1$ in estimating luminosities photometrically, there is a significant difference between M_{evo} and M_{puls}.

The apparent difference between M_{puls} and M_{evo} has been interpreted as an indication of: 1. significant mass loss at some point between the main sequence stage and the Cepheid stage (Cogan 1970); 2. errors in pulsation calculations (Rodgers 1970); 3. uncertainties in evolutionary calculations, particularly as these are influenced by uncertainties in opacities (Fricke et al 1971, 1972), and 4. a systematic error in the determination of the distance scale, possibly due to an underestimate of the distance to the Hyades (Iben & Tuggle 1972).

Lauterborn, Refsdal, & Weigert (1971) investigate the behavior of a model of initial mass $M = 5M_\odot$ which is not permitted to lose mass from the surface until it reaches the giant branch, just before the ignition of helium. An arbitrary reduction in mass is made at this point, and subsequent evolution proceeds at a new constant mass. Without mass loss, the blue limit reached during core helium burning extends well into the Cepheid instability strip and beyond. When mass is removed, the blue limit recedes to lower surface temperatures. A 10% reduction in mass is sufficient to bring the blue limit all the way out of the instability strip and over to the giant branch. Only when over half of the initial mass of the model is removed does the blue limit swing back into the region of the instability strip, in agreement with a study by Forbes (1968).

One might infer that, if a star is to pass through the Cepheid instability strip during core helium burning, it must lose either very little mass or very much. It is probable, however, that further calculations for higher initial masses and for different compositions will show that: 1. modest reductions in mass at the red giant tip will simply alter the location of the second helium-burning band in such a way that it crosses the relevant instability strip at a higher luminosity than would be the case in the absence of mass loss; and 2. small adjustments in composition parameters will bring the band back to its initial position. This must be the case, since some mass loss surely occurs during the first phase of core helium burning when a star spends a considerable fraction of its total helium-burning lifetime as a red giant.

Thus, extant calculations do not exclude the possibility that stars lose almost a third of their initial mass, as implied by (30) with $\delta \log L \approx 0$, and one must turn to the observations for arbitration. Van Altena (1973) has shown that the convergent point technique for estimating the Hyades distance gives a distance modulus that is smaller by about 0.2 mag than that given by all other methods. If one adopts the modulus given by the other methods, $\delta \log L(\mathrm{min}) \approx +0.08$ and one may argue that there is no longer any observationally based reason for supposing that M_{puls} and M_{evo} are significantly different from one another.

Given the lack of evidence for significant mass loss, one may eliminate mass between the results of evolutionary and pulsation calculations to derive a purely theoretical relationship between L, T_e, and P that may be used to estimate the distance to any Cepheid in the Galaxy. This relationship is composition dependent and, at present, only enough information is available to construct it properly for one set of composition parameters: $X \approx 0.7$, $Z \approx 0.02$. For this set (Iben & Tuggle 1972) we have

$$M_{\mathrm{BOL}} \approx -0.96-3.76 \log P-13.0(\log T_e-3.77) \tag{31}$$

Assuming that the chosen composition parameters are appropriate for most Cepheids in the Galaxy and adopting conversions between T_e and $B-V$ and between M_V and M_{BOL} that are thought appropriate for these Cepheids, one has finally (Iben & Tuggle 1972)

$$M_V \approx -2.61-3.76 \log P+2.60\,(B-V) \tag{32}$$

4.4 *The Development of an Electron-Degenerate Carbon-Oxygen Core*

Following the exhaustion of central helium, all stars of intermediate mass are found again on the giant branch, where helium burning in a thick shell provides most of the surface luminosity (Points G to K in Figure 5). As each model climbs upward along the second giant branch, convection extends inward until, in the more massive models of the intermediate-mass range (say $M \gtrsim 5M_{\odot}$), it reaches the hydrogen-helium discontinuity marking the location of the now defunct hydrogen-burning shell. As the base of the convective envelope moves in still deeper, helium is convected toward the surface and hydrogen is convected inward (Kippenhahn et al 1966). Before the inward march of convection is halted, the increase in the surface He/H ratio may be considerable. For example, in a $7M_{\odot}$ model there is a 40% increase in this ratio before the hydrogen shell is reignited and the base of the convective envelope is forced outward in mass (Iben 1972).

In stars more massive than about $10M_{\odot}$ (Paczynski 1970, 1971a), the mass internal to the base of the convective envelope does not decrease below $1.4M_{\odot}$ and the central portion of the star does not become electron degenerate before carbon or oxygen is ignited in this central region. In stars less massive than about $8M_{\odot}$ the inward march of convection is halted when the helium-burning shell approaches to within about $0.05M_{\odot}$ of the hydrogen-helium discontinuity, whereupon hydrogen is reignited. Once hydrogen burning is reestablished, the two nuclear-burning shells are separated by about 0.002–$0.005M_{\odot}$, and the mass in the carbon-oxygen (C-O) core is less than $1.4M_{\odot}$. The smaller the mass of the star, the smaller is the mass of the C-O core when hydrogen is reignited. Within the C-O core, cooling first by electron conduction and then by postulated neutrino losses (Weigert 1966) forces the central portion of the core into the region of relativistic electron degeneracy.

Once electrons in the core have become degenerate, the subsequent interior development is considerably influenced by the rate at which neutrino losses occur (compare Kippenhahn et al 1965, 1966 with Weigert 1966) via processes that might be viewed almost as fabrications out of whole cloth. Loss rates that are currently in use (e.g., Beaudet et al 1967, Festa & Ruderman 1969) follow from a choice of the Hamiltonian for weak interactions (Feynman & Gellmann 1958), which is certainly elegant but is by no means established by experiment. Other elegant formulations exist (Weinberg 1967, 1972) that lead to different rates (Dicus 1972), but these, too, are not established experimentally.

In the absence of the postulated neutrino losses, electron-degenerate C-O cores would still form. This conclusion follows from the inefficiency of converting work into thermal energy under degenerate conditions and from the efficacy of electron conduction in carrying out energy under such conditions. However, the maximum mass for the formation of such cores would be considerably reduced in the absence of neutrino losses at the postulated rates. The work of Kippenhahn et al (1965, 1966) shows that the maximum mass would lie well below $5M_{\odot}$.

Just as in the case of low-mass stars that develop common helium cores, so, too, stars of intermediate mass form common C-O cores once core electrons become sufficiently degenerate. The phenomenon of convergence toward a common C-O core has been demonstrated by Paczynski (1970, 1971a) and by Uus (1970) for the case in which neutrino losses occur at currently postulated rates. In contrast to the case of low-mass stars, energy losses via neutrino production processes dominate energy losses via electron conduction. In fact, a good approximation to the common path of central density and temperature in Figure 4 can be achieved simply by equating the neutrino loss rate at the center with the difference between the rate at which work is being done on matter at the center by surrounding matter in the contracting core and the rate at which the particle kinetic energy there is being increased (Barkat 1971, Paczynski 1971a, and Arnett 1971a).

Over most of the region between the center and the maximum in the temperature distribution within the C-O core (see Figures 7 and 9), neutrino losses nearly equal the difference between the fraction of gravitational potential energy that is converted into local work and the fraction that is converted into internal kinetic energy. In contrast to the situation in electron-degenerate helium cores, the temperature gradient controls the flow of energy via conduction and radiation, rather than vice versa. An example of the distribution of postulated neutrino loss rates is given in Figure 1 of Iben (1972), which also illustrates that the concept of common cores is a bit of an oversimplification.

The distributions of carbon and oxygen in the inner parts of the core are a function of the initial mass of the star. The more massive the star, the lower the abundance of carbon at the center will be when helium is exhausted there (see, e.g., Vidal et al 1971, Arnett 1971b). Furthermore, the larger the initial mass of the star, the larger the central convective region during core helium burning will be, and the further out in mass will the first discontinuity in the C-O abundances occur.

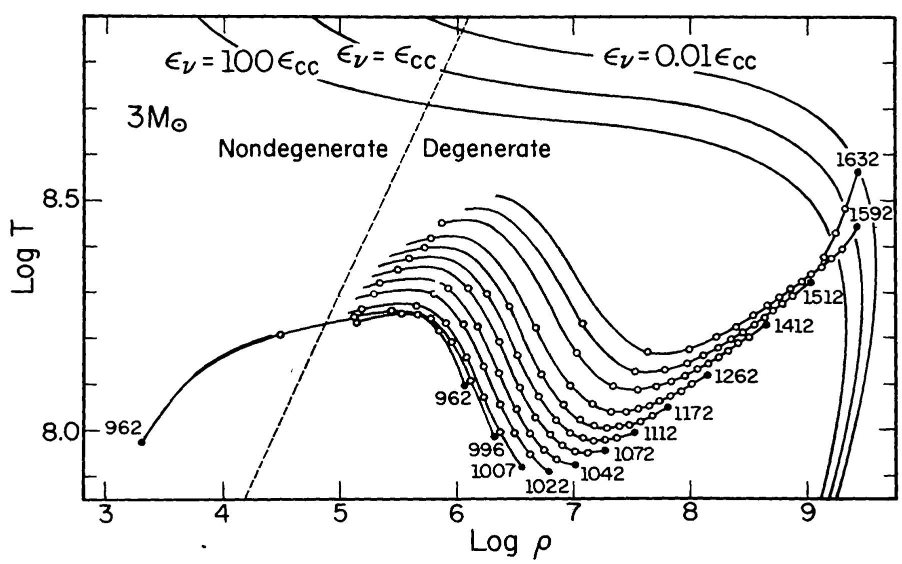

Figure 7 Density and temperature within the growing carbon core of a $3M_\odot$ model (Paczynski 1971a). Larger numbers correspond to larger electron-degenerate cores.

Where and under what circumstances in the electron-degenerate core carbon or oxygen will be ignited depends not only on the uncertain distribution of carbon and oxygen, but on a number of additional factors in which there are also large uncertainties. Following Weigert (1966), Arnett (1969), and Paczynski (1971a), an approximate criterion for carbon ignition may be obtained by equating a carbon-burning rate with a neutrino loss rate. The ignition curve in Figure 4 has been obtained by 1. setting the carbon abundance by mass equal to 0.5 in a burning rate based on the experimental work of Mazarakis & Stephens (1972), 2. including the effect of electron screening on the carbon-carbon reaction rate in the approximation of Dewitt et al (1973), and 3. assuming neutrino loss rates given by Beaudet et al (1967) and by Festa & Ruderman (1969). At low densities, electron screening and the electron-neutrino bremsstrahlung process play no role. At high densities, both processes are important. In fact, it is the enormous enhancement of the carbon-burning rate by electron screening at high densities that is the primary reason for the high density limit to the ignition curve.

The uncertainties in the carbon-burning rate and in the electron-screening factor do not appear to be terribly important in determining the position of the ignition curve. This is because of the very steep temperature and/or density dependences of all of the relevant rates. The high density limit to the ignition curve would not seem to be uncertain by over a factor of 5 or so in density, provided the postulated neutrino rates are correct and provided the abundance of carbon is not excessively small ($\lesssim 0.01$). Uncertainties in screening do not appear to affect the ignition density by more than a factor of 2 or so (Graboske 1973). As shown in Figure 7 (from Paczynski 1971a), a variation by a factor of one hundred in either the neutrino loss rates or in the carbon-burning rate would not alter appreciably the general location of the ignition curve.

Carbon ignition will occur in that portion of the core which first crosses the appropriate ignition line. In stars more massive than about $10M_\odot$ (Paczynski 1970, 1971a), ignition occurs first at the stellar center, before electron degeneracy becomes important and before neutrino losses can invert the temperature gradient near the center. In stars of mass around $9M_\odot$, ignition occurs off center in a weakly degenerate region, after the central portion of the star has already begun its descent away from the ignition curve (Iben 1973a). It is expected that a thermal runaway will occur in layers successively closer to the center, with degeneracy being lifted in each layer in turn. The anticipated development is analogous to what happens in low-mass stars when helium is ignited off center in an electron-degenerate helium core (Thomas 1967). The net final result will be quiescent carbon burning under nondegenerate conditions. In stars less massive than about $8M_\odot$, it would appear that carbon ignition occurs again at the center along the high density boundary of the ignition curve (Arnett 1969, Paczynski 1970, 1971a). An example of central ignition is shown in Figure 7.

The lower mass limit for carbon ignition depends on how much mass is lost from the surface during the final giant phase. Paczynski (1970, 1971a) argues that stars initially less massive than $3.5M_\odot$ will lose mass to such an extent via a radiation-pressure driven process that the remnant core will be less massive than $1.4M_\odot$ before any point in the C-O core reaches ignition temperatures. The radiation pressure mechanism may not be as effective as anticipated (e.g., Joss et al 1973), but there are other instabilities that develop that may lead to ejection of mass in sufficient quantities to leave a remnant less massive than $1.4M_\odot$. Finally, there is the likelihood of carbon grain formation at the surface during the last red-giant stage and extensive mass loss brought about by expulsion of these grains.

4.5 *Thermal Relaxation Oscillations, Mixing, and Surface Abundances*

Thermal relaxation oscillations in stars of intermediate mass create a rhythm quite different from that produced by such oscillations in low-mass stars. Instead of being separated by hundreds of thousands of years, as in the case of low-mass stars, thermal pulses are separated by only a few thousand years and are not broken up into subpulses. This is shown in Figure 8 (from Weigert 1966), which is to be contrasted with Figure 2. At maximum, the rate of energy production by helium burning is about double the maximum in the luminosity function. During the first few thermal pulses, beginning just after the hydrogen-burning shell is re-ignited, the magnitude of L_{HE} (peak rate of energy production by helium burning) increases dramatically. For example, in a $7M_\odot$ model (Iben 1974), L_{He} takes on successive values: $L_{\mathrm{HE}}/10^4 L_\odot \sim 1.2$, 5.6, 14.7, 25.6, 36.3, 53.9, 75.2, 99.5. In contrast, at its maximum output during the interpulse phase, the hydrogen-burning shell produces energy at the rate of only about $3.5 \times 10^4 L_\odot$.

It would be interesting to follow the development of the thermal pulses as the C-O core grows in mass to $1.4M_\odot$. However, brute force calculations are highly impractical; about 10^6 time steps would be required to follow pulses as core mass grows from, say, 0.9 to $1.4M_\odot$. Clearly, some short cuts must be taken. Paczynski (1970, 1971a) places both burning shells in a static envelope, assuming that entropy changes, dS/dt, can be approximated by $-(\partial S/\partial m)\dot{m}$ where $\dot{m}$ is the rate at which the hydrogen-burning shell converts mass m into helium. This procedure damps out the thermal instability completely and may permit a fair representation of the evolution of the C-O core, provided that the thermal instability in a real star does not reach sufficient amplitude to affect mean core behavior. Uus (1970), Eggleton (1973), and Sugimoto & Nomoto (1973) have devised alternate procedures.

Sugimoto & Nomoto follow through two thermal pulses, one each for core masses of 1.07 and $1.39M_\odot$, finding $L_{\mathrm{He}} \sim 10^7 L_\odot$ in both cases. One might infer that L_{He} reaches an asymptotic value. However, it takes several pulses to build up to a local steady state and the actual pulse amplitude for these masses may be much larger.

Very important for its possible observational consequences is the nature and extent of mixing during the progress of both the thermal pulse and the relaxation phase. Extant calculations show that the convective shell that builds up within the helium-burning region reaches almost to the hydrogen-helium discontinuity at the peak of the pulse. When the pulse dies down and the rate of helium burning is reduced to nominal values, the base of the convective envelope extends inward, reaching its greatest average penetration when the hydrogen-burning shell reaches its peak in power output.

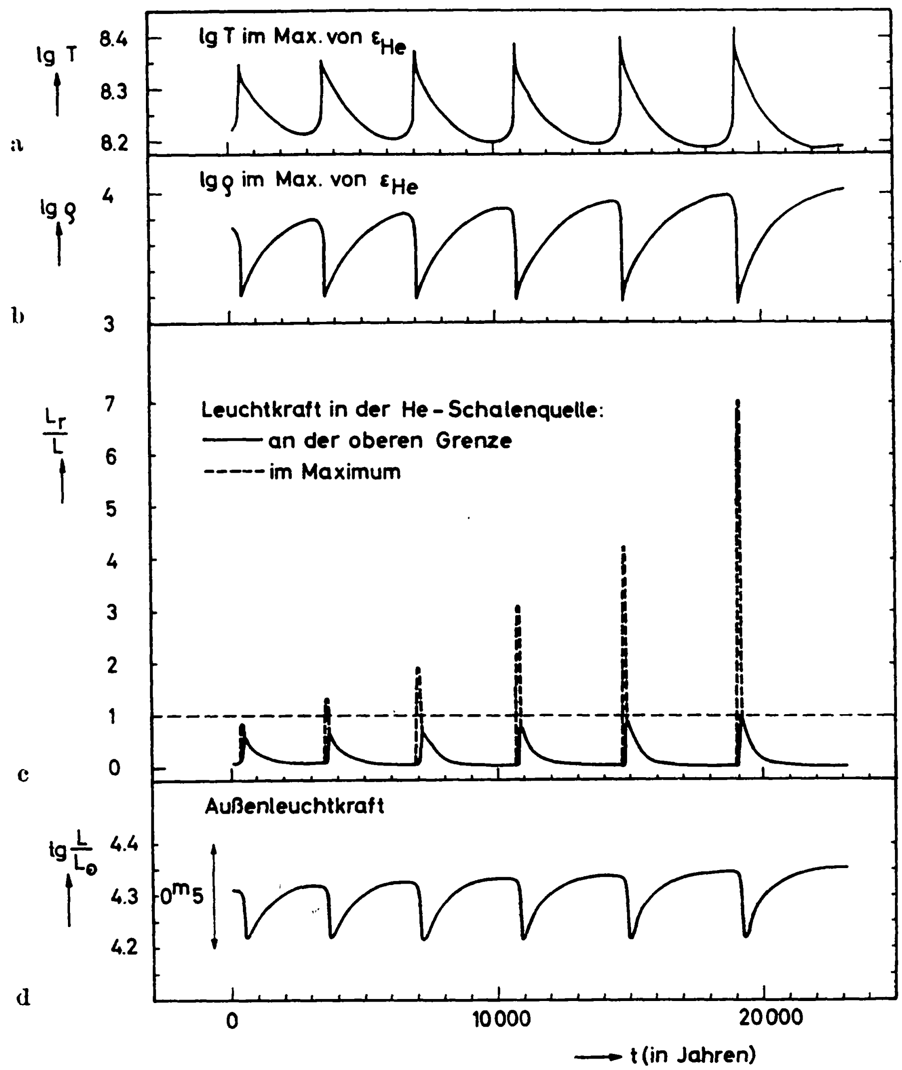

Figure 8 Relaxation oscillation in a model of intermediate mass (Weigert 1966). The symbols are defined as follows: (a) $\log_{10} T$ (temperature) at the position of the maximum in ε_{He} (energy generation rate by helium burning); (b) $\log_{10} \rho$ (density) at maximum in ε_{He}; (c) luminosity (in units of surface luminosity) in the helium shell. The solid curve gives the luminosity at the outer edge of the shell and the dashed curve gives the maximum luminosity; (d) $\log_{10} (L/L_\odot)$ where L is the luminosity at the surface. The abscissa is time elapsed in years.

Whether or not the outer edge of the convective shell will extend into the hydrogen-rich region has not yet been determined. How deep envelope convection penetrates into the hydrogen-burning shell depends on the treatment of convection in the outer regions of the envelope, where the temperature gradient is highly superadiabatic (e.g., Paczynski 1970, 1971a, Sugimoto 1971, Uus 1970, 1971). For a thorough review of this problem, see Nomoto & Sugimoto (1973).

The phase relationship between the convective shell and the convective envelope

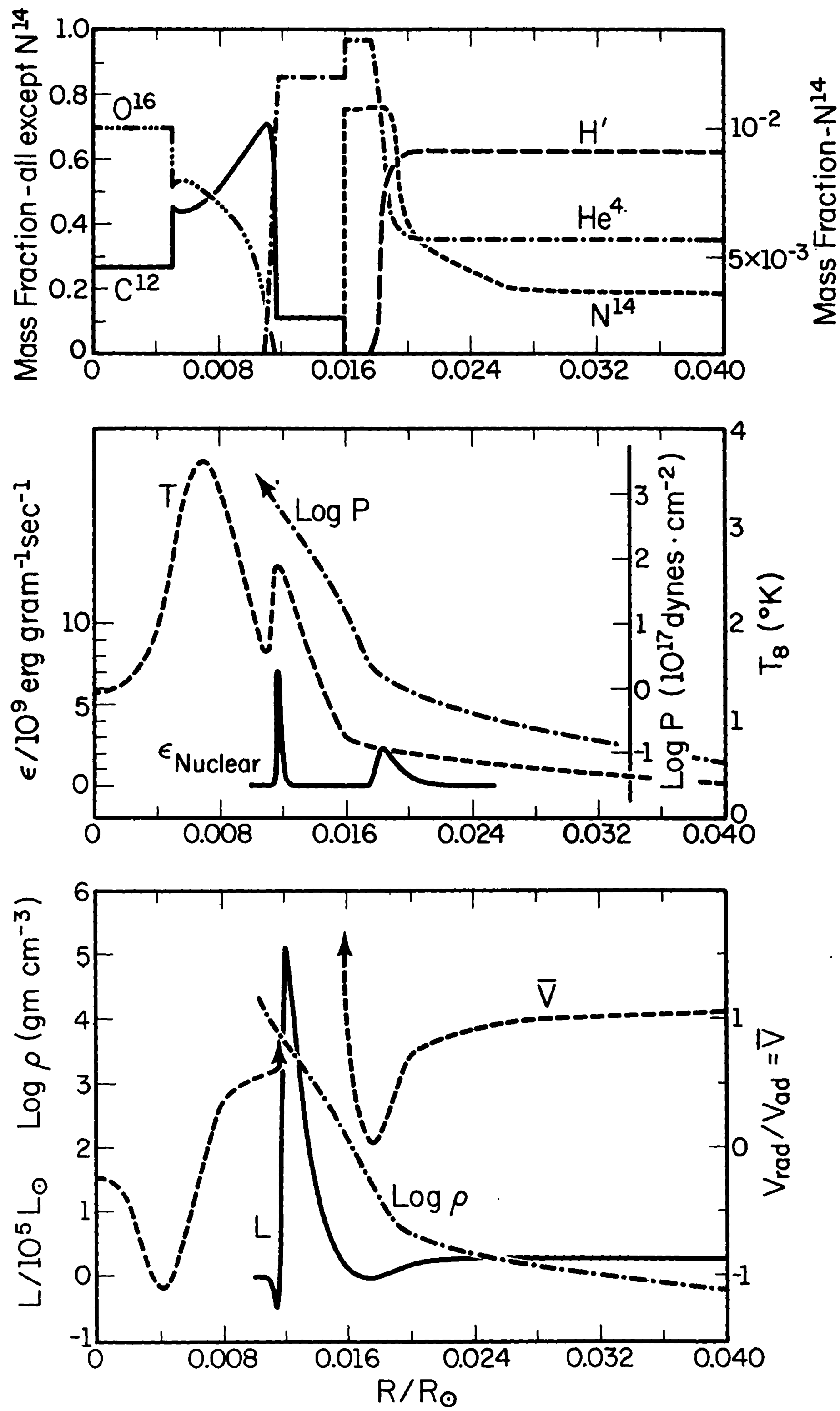

Figure 9 Distribution of composition parameters and state variables within a $7M_{\odot}$ model star at the peak of a thermal pulse (Iben 1974).

seems to be well established. As the outer edge of the convective shell moves outward, so does the base of the convective envelope. This is because, as the helium-burning rate increases, almost all of the excess energy goes into expanding matter within and on either side of the burning region. The hydrogen-helium discontinuity is pushed out to lower temperatures and densities and the hydrogen-burning rate drops. The decrease in flux that escapes from the two burning regions forces the base of the convective envelope to move outward in both mass and radius.

This picture may be modified for later pulses if mixing takes place across the hydrogen-helium discontinuity during these later pulses. A number of schemes have been invented to describe the possible consequences of mixing across the hydrogen-helium interface (Ulrich & Scalo 1972, Scalo & Ulrich 1973, Smith et al 1973, Ulrich 1973). Unfortunately, no one has yet demonstrated that these schemes can be initiated. What we do know with some certainty is illustrated in Figure 9, where details are shown in the nuclear burning region of a $7M_{\odot}$ model at the peak of an early flash (Iben 1974).

Almost all of the energy generated by helium burning is used up in expansion. Within the convective shell, all N^{14} has been converted into O^{18} or Ne^{22}. The separation in mass between the outer edge of the convective shell and the base of the hydrogen-helium discontinuity is only about $10^{-4}M_{\odot}$ in mass, but the separation in radius is over $10^{-3}R_{\odot}$. The fact that the two boundaries are separated by only two pressure scale heights suggests that convective overshoot might be carrying some matter through the formally radiative region into the hydrogen-rich region.

With each successive thermal pulse, the outer edge of the convective shell creeps ever closer to the hydrogen-rich region. It is possible that, after a sufficiently large number of pulses, this edge will extend into the hydrogen-rich region where protons are more abundant than N^{14} (a neutron poison). Then, hydrogen and N^{14} will be convected downward into a region where C^{12} outnumbers both the invading protons and the invading N^{14}. The consequence will be efficient *s*-processing. At the same time, C^{12}, O^{16}, Ne^{22}, and newly created *s*-process elements will be convected upward into the now-blurred hydrogen-helium interface.

When the thermal pulse has run its course and the convective shell has disappeared, the blurred hydrogen-helium transition region will fall back to high enough temperatures and densities for hydrogen to reignite. Finally, the base of the convective envelope may then penetrate far enough into the outer edge of the hydrogen-helium transition region to convect outward the newly formed elements found in this region. Thus, it may not be essential that direct mixing contact between the envelope convective region and the convective shell be established in order to bring newly formed *s*-process elements to the surface. It is quite possible that the production of exotic elements is "180°" out of phase with the mixing of these elements to the surface.

4.6 *Carbon Detonation or Carbon Fizzle?*

Arnett (1968, 1969) and Rose (1969) were the first to suggest that the ignition of carbon in single stars of intermediate mass might trigger a supernova event.

Arnett shows that, if the ignition of carbon at the center of the helium-exhausted core of such stars develops into a detonation wave, then this wave will impart to all matter in the star velocities greater than the local escape velocity. Arnett (1969), Buchler et al (1971), and Bruenn (1972) have examined the self-consistency of the solutions for the detonation wave. Within and behind the detonation front, temperatures may rise high enough for all matter to be processed into the region of iron-peak elements, as shown explicitly by Arnett, Truran, & Woosley (1971) and Bruenn (1971). This implies a release of about 2×10^{51} erg of nuclear energy into the initially electron-degenerate core. Since only about half of this energy is required to overcome the binding energy of the core and to impart greater than escape velocities to all of the matter in the star, there is ample energy remaining to account for the 10^{49}–10^{50} erg of visible light that emanates from supernovae.

However, the expulsion of $1.4M_\odot$ of iron-peak elements presents difficulties for understanding abundance distributions both in the gas in the immediate vicinity of known supernova events, where only the abundance of helium seems abnormally high (Woltjer 1958, Peimbert & van den Bergh 1971), and in the galactic gas or at the surfaces of young stars recently born out of this gas. Another embarrassment is provided by the pulsars, which are reasonably well established as neutron star remnants of supernovae (see Ruderman 1972). Finally, the amount of energy needed to account for the state of ionization of matter surrounding at least one pulsar (Vela) could possibly be as large as 10^{53} erg (see Maran & Brandt 1974). Thus, the carbon detonation model of supernovae runs into difficulty on three grounds; too much iron, no remnant, and possibly not enough energy emitted.

The problem is compounded by the fact that the birth rate of stars in the mass range 3–$8M_\odot$, all of which develop a C-O core of mass $1.4M_\odot$ unless surface mass loss diminishes them below $1.4M_\odot$, is at least comparable to the inferred rate at which supernovae of type II occur. This estimate of birth rate is based on an estimate of 2×10^4 Cepheids in the Galaxy (Kukarkin & Paranego 1963), an estimate of the masses of known Cepheids (e.g., Iben & Tuggle 1972), and an estimate of the time spent in the Cepheid instability strip by a core helium-burning model (e.g., Hofmeister 1967).

How does one solve the dilemma? The simplest solution is to suppose that most of the potentially offending stars evaporate before they reach the offensive state, as suggested by Woolf (1973). There are enough uncertainties in the observations of cool supergiants and in the analysis of their spectra to make it premature to accept this solution as anything other than an entertaining possibility.

It is, however, probable that, because of substantial mass loss during the double shell-source stage, stars of intermediate mass make an important contribution to the enrichment of the galactic gas in several of the lighter elements, even if these stars do not divest themselves of all but a remnant mass of less than $1.4M_\odot$. Certainly such stars contribute to an enrichment in He^3, He^4, N^{14}, C^{13}, and possibly Li^7 (Cameron & Fowler 1970, Iben 1972, 1973b). If there is a production of *s*-process elements during thermal pulses and if these elements are carried to the surface following each pulse, then stars of intermediate mass should also contribute to the interstellar medium new *s*-process elements, as well as new C^{12} and Ne^{22}, and possibly new O^{16} and O^{18}.

Another proposed solution is that electron capture on several products of the nucleosynthesis that follows carbon ignition reduces the electron pressure enough for the central portions of the star to implode. Wheeler et al (1970) show that an implosion cannot be initiated unless detonation begins at densities greater than 6×10^9 gm/cm^3. Bruenn (1972) shows that, if central detonation begins at densities in the range $6 \times 10^9 \rightarrow 2 \times 10^{10}$ gm/cm^3, the inner part of the stellar core contracts for a time but ultimately disperses. Only for carbon ignition densities greater than 3×10^{10} gm/cm^3 will the inner portion of the core ultimately implode toward neutron star densities. Calculations by Wheeler et al (1973) give similar results. For core implosion to follow detonation, then, it is necessary to delay carbon ignition until the central density exceeds 3×10^{10} gm/cm^3, a rather remote possibility (Barkat et al 1972).

For sufficiently small abundances of central carbon, say $X_{12} \gtrsim 0.01$, ignition will not lead to detonation, simply because not enough energy will be released to raise temperatures far enough to ignite additional fuels. One might then expect a quiet phase of carbon burning at the center, followed by further contraction of central regions as carbon burns in a shell that works its way out through the core. One might anticipate that the ignition of oxygen at the center would occur at sufficiently high densities ($\rho > 3 \times 10^{10}$ gm/cm^3) that detonation and implosion might result in both a condensed and an extended remnant. Given the uncertainties in the effective cross-section factor for the $C^{12}(\alpha, \gamma)$ O^{16} reaction, this solution to the problem would appear to be as likely as any, were it not for the fact that it would present us with another problem: how to make carbon in the Universe.

A low abundance of carbon at the centers of the helium-exhausted cores of stars of intermediate mass implies an even lower final abundance of carbon achieved during helium burning in more massive stars. Thus, massive stars could not be the major source of carbon in the intergalactic gas. Nor could stars of intermediate mass be the major source of carbon unless carbon left behind by the helium-burning shell is at an abundance considerably larger than that of the carbon left behind at the center, or unless some of the carbon formed in the convective shell during a thermal pulse is mixed to the surface, where simultaneously substantial mass loss is occurring.

Perhaps the most promising solution to the dilemma has been proposed by Paczynski (1972b, 1973a), who suggests that energy losses from an Urca shell (Tsuruta & Cameron 1969) might drain off the energy created by the carbon-burning reactions near the center and thus prevent detonation. The idea is that, when carbon is first ignited, it forces convection to spread out from the center. As the edge of the growing convective core passes through a density such that the electron Fermi energy is equal to the threshold for electron captures on the proton rich member of an Urca pair, N^+ and N^-, energy in the form of a neutrino will be lost as the N^+ elements are convected from the low-density to the high-density side of the Urca shell, where they capture electrons. Similarly, energy in the form of antineutrinos will be lost as the N^- elements are convected across the Urca shell from high to low densities and emit electrons.

Bruenn (1973) has pointed out that the rearrangement of electrons in the Fermi sea that takes place following an electron capture or an electron emission can lead to local heating as well as to cooling. Thus, the Urca process has a more complex local influence than a simple loss of internal free energy equal in magnitude to the energy taken off by the neutrinos. Bruenn (1973) argues that, as a consequence of the heating effect, an Urca shell will not prevent central detonation of carbon, whereas Paczynski (1973a) argues that the situation is not clear cut.

Adopting a global point of view, it seems plausible that, if the rate of electron captures and the rate of electron emission are equal, when averaged over the entire Urca shell, then the Urca process must always rob internal energy from the star and, whatever the details of the conversion, nuclear energy created by carbon burning will be converted into this escaping energy. Couch & Arnett (1973) have examined the consequences of assuming that the energy generated by carbon burning exactly balances Urca neutrino losses in a convective core whose outer edge extends slightly beyond the Urca shell for the pair Na^{23}–Ne^{23}. This pair is chosen because it is an abundant (few percent by mass) product of carbon burning. Within the convective region the production of the neutron rich isotopes Na^{23}, Ne^{23}, Mg^{25}, and Mg^{26} increases the electron molecular weight μ_e there to such an extent that the effective Chandrasekhar mass of the carbon-oxygen core, $\langle 5.76 M_{\odot}/\mu_e^2 \rangle$, becomes smaller than the mass contained in the core when carbon was ignited. Because of the reduction in the number of pressure-supplying electrons per nucleon, the core begins to contract on a dynamic time scale. Paczynski & Ergma (1973) present results similar to those of Couch & Arnett and suggest that the dynamic collapse might lead to a neutron star of mass near $0.4 M_{\odot}$, the mass of the convective region at the onset of collapse. They argue further that a detonation front may form at the point where carbon is ignited in a shell at $M \approx 0.4 M_{\odot}$ and that, on passing through the star, this front may impart to matter velocities greater than escape.

Another picture for the expulsion of the outer portions of the star has been proposed by Ostriker & Gunn (1972), who suggest that, however a dynamic collapse of the interior portions of a star is initiated, a neutron star will be formed in a time small compared to the time required by the envelope to fall very far toward the center. They suggest that the beam of electro-magnetic waves emitted by the spinning neutron star will fill up the cavity between the neutron star and the envelope and that the resulting radiation pressure at the base of the envelope will drive off this envelope.

5 EVOLUTION OF MASSIVE STARS

No attempt will be made to review properly the recent literature on the evolution of massive stars, defined here as stars which do not develop a strongly electron-degenerate core until all exoergic reactions have run to completion at the center. For literature reviews see Stothers & Chin (1969), Stothers (1972), Ruben (1969), Dallaporta (1971), and Massevich & Tutukov (1973).

Until recently it was commonly believed that a nuclear-driven pulsational instability (Ledoux 1941), which occurs in main sequence models of mass greater than some critical mass M_{crit} (about $60M_{\odot}$ for population I: Schwarzschild & Härm 1959), was responsible for the dearth of extremely massive stars in the Galaxy. The thought was that the pulsation amplitude might grow without limit and mass would be lost from the surface until the total stellar mass was reduced below the critical mass for pulsational instability.

The dynamical calculations of Appenzeller (1970), Ziebarth (1970), Talbot (1971), and Taylor & Papaloizou (1973) show explicitly that the mechanical energy of pulsations is converted into running shock fronts that expel matter from the surface. The damping introduced by this conversion limits the pulsation amplitude to a finite value. Mass loss rates are estimated to be sufficiently small ($\sim$few $\times 10^{-5}M_{\odot}$/yr) that a star considerably more massive than M_{crit} will remain more massive than M_{crit} for its main-sequence lifetime.

Thus, the paucity of main sequence stars more massive than M_{crit} must be due to accidents that occur at birth. Larson & Starrfield (1971) examine several effects that might impede the formation of very massive stars and conclude that the dominant deterrent is the tendency for a stellar core to form at the center of a larger, contracting cloud. The radiation from the stellar core forms an HII ionization front that spreads out into the still infalling outer regions of the cloud, reversing the direction of matter through which it passes. The maximum mass of the stellar core is estimated by equating the time to form the core to the main sequence lifetime of the core. Clearly, this maximum mass is a function of initial conditions but, for "reasonable" choices of initial conditions, it turns out to be near 60–$120M_{\odot}$.

As in the case of stars of intermediate mass, there are two distinct regions in surface temperature where considerable time may be spent during core helium burning: one "red" region of nearly constant temperature near the giant branch and another "blue" region with a larger range in T_e centered at a point between the main sequence and the giant branch. Both regions are at essentially the same luminosity, a consequence of the dominant contribution of radiation pressure to the total pressure. How much time is spent in each region is related to the hydrogen profile through which the hydrogen-burning shell progresses while helium still remains at the center. This profile is most conspicuously influenced by whether or not a (truly) convective shell appears toward the end of the main sequence phase and by the depth which envelope convection reaches while the star is on the giant branch.

The occurrence or non-occurrence of a convective shell depends upon the choice of criterion for stability against convection (e.g., Stothers & Chin 1968, 1969, 1973, Chiosi & Summa 1970, Simpson 1971, Stothers 1972, Robertson 1972, Barbaro et al 1972, Ziolkowski 1972, Varshavsky 1972). If the convective shell does occur, then considerable time during core helium burning is spent in the blue region, independent of whether helium ignition first occurs as the model passes into the blue region or whether it first occurs in the red region.

If the convective shell does not appear, then a model spends all of its core helium-burning phase as a red supergiant unless the hydrogen-burning shell reaches the discontinuity in the hydrogen profile established by the convective envelope at its greatest inward penetration. If the hydrogen-burning shell reaches this discontinuity, then the model will jump to the blue region and remain there for the greater portion of the remaining phase of core helium-burning (Ziolkowski 1972, Lauterborn, Refsdal, & Roth 1971, Kozlowski 1971, Robertson 1972, Stothers & Chin 1973).

Whatever the eventual consensus will be regarding the stability criterion for convection and regarding the parameters that determine the depth of penetration of convective envelopes along the giant branch, the observations suggest that a major fraction of the core helium-burning lifetime of a population I star in our Galaxy is spent as a blue supergiant, the "blueness" of the "blue" region increasing with stellar mass (Humphreys 1970).

During all phases more advanced than core helium burning, massive stars remain on the red supergiant branch (e.g., Stothers & Chin 1969, Varshavsky Tutukov 1972, 1973). Apart from the question of surface mass loss (e.g., Bisnovatyi-Kogan & Nadyozhin 1972, Zytkow 1973, Woolf 1973) all of the action of potential observational significance takes place in the deep interior, where successive layers of more complex elements are deposited by successively more refractory nuclear fuels.

Evolution up through the final exoergic nuclear-reaction phase—namely, the formation of iron-peak elements—has been carried out by Sugimoto (1970, 1971) and by Arnett (1972, 1973a) in models that are initially pure helium and by Ikeuchi et al (1971, 1972) in models that are initially pure carbon and oxygen. These studies show that, in contrast to the situation during phases earlier than core carbon burning, the occurrence or non-occurrence of neutrino losses at rates suggested by elegant but unconfirmed theories play an essential rather than a peripheral role in determining the sequences of heavy elements that are successively formed.

Unfortunately, the omission of a hydrogen envelope or of a helium-hydrogen envelope precludes all study of mass loss (probably significant) and prevents any estimate of how observable features are affected by the inward penetration of a convective envelope (both as regards surface abundance changes and as regards enrichment of the galactic gas by products of interior processing). In models without a hydrogen envelope, an onion-ring distribution of elements is built up (e.g., Arnett 1973b). When a hydrogen envelope is considered, the inward penetration of convection can, under appropriate conditions, smear the onion-ring like arrangement out over the entire convective region (e.g., Nomoto & Sugimoto 1973). If mass loss from the surface is important, as might be expected in cool, highly luminous red supergiants, this means that elements newly formed within the bowls of a massive star are brought to the surface and injected into the interstellar medium before any dynamic fate befalls the star, whatever combination of implosion and explosion this fate may be.

The details of the final, dynamic event have yet to be elucidated. However, current indications are that, once densities and temperatures in the growing iron core cross the Fe, He, n phase-transition curve, iron will be photodecomposed into helium nuclei and neutrons. The energy for photodecomposition will be provided by the release of gravitational potential energy from a core whose collapse is initiated by the underpressure that accompanies the reduction in the number of electrons per nuclei (see Colgate 1971).

Whether the collapsing core will be joined by the infalling envelope to form a black hole, or whether the envelope will be ejected by radiation pressure from a centrally formed neutron star, by the detonation of some remaining nuclear fuel, or by neutrino deposition in the envelope has yet to be decided. As a consequence of surface mass loss, there may be so little envelope matter left by the time an "iron" core is formed, that no central implosion occurs at all.

ACKNOWLEDGMENTS

It is a pleasure to thank many members of the University of Illinois faculty for discussions of topics touched upon in this review. Particular thanks go to Gordon Baym, Michael Howard, Fred Lamb, Susan Lamb, David Pines, Jim Truran, and Roy Tuggle. Thanks also to Vittoria Caloi, Vittorio Castellani, P. Giannone, Franco Pacini, and Alveo Renzini for organizing and contributing so much to the success of a conference on globular cluster stars held in Frascati, June 1973, where many of the ideas discussed in III were clarified. Thanks to Olin Eggen and Alex Rogers for organizing a conference on stellar evolution and instability held in Canberra, August 1973, where material covered in III and IV was exhaustively discussed. Further thanks to David Arnett and David Schramm for organizing a conference on explosive nucleosynthesis held in Austin, March 1973, where much of the material discussed in IV and V was clarified. Finally, thanks to the editorial staff of the Annual Reviews for its forbearance.

Literature Cited

Aizenman, M. L., Demarque, P., Miller, R. H. 1969. *Ap. J.* 155:973
Appenzeller, I. 1970. *Astron. Ap.* 5:355; 9:216
Arnett, W. D. 1968. *Nature* 219:1344
Arnett, W. D. 1969. *Ap. Space Sci.* 5:180
Arnett, W. D. 1971a. *Ap. J.* 169:113
Arnett, W. D. 1971b. *Ap. J. Lett.* 170:L43
Arnett, W. D. 1972. *Ap. J.* 176:681, 699
Arnett, W. D. 1973a. *Ap. J.* 179:249
Arnett, W. D. 1973b. *Explosive Nucleosynthesis*, ed. D. N. Schramm, W. D. Arnett, p. 236. Univ. Texas, Austin
Arnett, W. D., Truran, J. W., Woosley, S. E. 1971. *Ap. J.* 165:87
Bahcall, J. N., Sears, R. L. 1972. *Ann. Rev. Astron. Ap.* 10:25
Baker, N. H., von Sengbush, K. 1969. *Mitt. Ast. Ges.* No. 27
Barbaro, G., Chiosi, C., Nobili, L. 1972. *Astron. Ap.* 18:186
Barkat, Z. K. 1971. *Ap. J.* 163:433
Barkat, Z. K., Buchler, J. R., Wheeler, J. C. 1970. *Ap. Lett.* 6:117
Barkat, Z. K., Wheeler, J. C., Buchler, J. R. 1972. *Ap. J.* 171:651
Beaudet, G., Petrosian, V., Salpeter, E. E. 1967. *Ap. J.* 150:979
Bisnovatyi-Kogan, G. S., Nadyozhin, D. K. 1972. *Ap. Space Sci.* 15:353
Böhm-Vitense, E., Szkody, P. 1973. *Ap. J.* 184:211
Bruenn, S. W. 1971. *Ap. J.* 168:203
Bruenn, S. W. 1972. *Ap. J. Suppl.* 24:283
Bruenn, S. W. 1973. *Ap. J. Lett.* 183:L125
Buchler, J. R., Wheeler, J. C., Barkat, Z. K. 1971. *Ap. J.* 167:465
Caloi, V. 1972. *Astron. Ap.* 20:357
Cameron, A. G. W. 1970. *Ann. Rev. Astron. Ap.* 8:179
Cameron, A. G. W. 1973. *Rev. Geophys. Space Sci.* 11:505
Cameron, A. G. W., Fowler, W. A. 1971. *Ap. J.* 164:111
Cannon, R. D. 1970. *MNRAS* 150:111
Cannon, R. D. 1973. *Frascati Conference on Globular Cluster Stars*
Cassinelli, J. P., Castor, J. I. 1972. *Ap. J.* 179:189
Castellani, V. 1973. *Frascati Conference on Globular Cluster Stars*
Castellani, V., Giannone, P., Renzini, A. 1969. *Ap. Space Sci.* 4:103
Castellani, V., Giannone, P., Renzini, A. 1971. *Ap. Space Sci.* 10:340, 355
Cayrel, R. 1968. *Ap. J.* 151:1005
Cayrel de Strobel, G., Delplace, A. M., Eds. 1972. *Age Des Etoiles*, Proc. IAU Colloq. No. 17 (Paris-Meudon Observatory)
Chiosi, C., Summa, C. 1970. *Ap. Space Sci.* 8:478
Christy, R. F. 1966a. *Ap. J.* 144:108

Christy, R. F. 1966b. *Ann. Rev. Astron. Ap.* 4:353
Cogan, B. C. 1970. *Ap. J.* 162:129
Colgate, S. A. 1971. *Ap. J.* 163:221
Couch, R. G., Arnett, W. D. 1973. *Ap. J. Lett.* 180:L101
Cox, J. P., Guili, R. T. 1968. *Principles of Stellar Structure*. New York: Gordon & Breach
Dallaporta, N. 1971. *Trieste Symposium on Supergiant Stars*
Davis, R. Jr. 1972. *Bull. Am. Phys. Soc.* 17: 527
Defouw, R. J., Siquig, R. A., Hansen, C. J. 1973. *Ap. J.* 184:581
Deinzer, W., Salpeter, E. E. 1964. *Ap. J.* 140:499
Demarque, P. 1967. *Ap. J.* 149:117
Demarque, P., Heasly, J. N. 1971. *MNRAS* 155:85
Demarque, P., Mengel, J. G. 1971. *Ap. J.* 164:317, 469
Demarque, P., Mengel, J. G. 1972. *Ap. J.* 171:583
Demarque, P., Mengel, J. G. 1973. *Astron. Ap.* 22:121
Demarque, P., Mengel, J. G., Aizenman, M. L. 1970. *Ap. J.* 163:37
Demarque, P., Mengel, J. G., Sweigert, A. 1972. *Ap. J.* 173:L27
Demarque, P., Mengel, J. G., Sweigert, A. 1973. *Ap. J.* 183:997
Devine, N. 1965. *Ap. J.* 142:824
Dewitt, H. E., Graboske, H. C., Cooper, M. S. 1973. *Ap. J.* 181:439
Dickens, R. J. 1971. *Ap. J. Suppl.* No. 187 22:249
Dickens, R. J., Wooley, R. 1967. *Roy. Observ. Bull.* 128:E255
Dicus, D. A. 1972. *Phys. Rev. D* 6(4):941
Dilke, F. W. W., Gough, D. O. 1972. *Nature* 240:262
Donn, B., Wickramasinghe, N. C., Hudson, J. P., Stecher, T. P. 1968. *Ap. J.* 153:451
Dyer, P. 1973. *Explosive Nucleosynthesis*, ed. D. N. Schramm, W. D. Arnett, p. 195. Univ. Texas, Austin
Edwards, A. C. 1970. *MNRAS* 146:445
Eggen, O. J., Sandage, A. 1969. *Ap. J.* 158:669
Eggleton, P. 1973. *MNRAS* 163:279
Ezer, D., Cameron, A. G. W. 1972. *Nature* 240:180
Faulkner, D. J. 1970. *Ap. J.* 162:513
Faulkner, D. J., Cannon, R. D. 1973. *Ap. J.* 180:435
Faulkner, D. J., Wood, P. R. 1972. *Ap. J.* 178:207
Faulkner, J. 1966. *Ap. J.* 144:978
Faulkner, J., Iben, I. Jr. 1966. *Ap. J.* 144:995
Fernie, D., Ed. 1972. *Variable Stars in Globular Clusters and in Related Systems*. Dordrecht, the Netherlands: Reidel
Festa, G. G., Ruderman, M. A. 1969. *Phys. Rev.* 180:1227
Feynman, R. P., Gellmann, M. 1958. *Phys. Rev.* 109:193
Finzi, A., Wolf, R. A. 1971. *Astron. Ap.* 11:418
Forbes, J. E. 1968. *Ap. J.* 153:495
Fowler, W. A. 1972. *Nature* 238:24
Fowler, W. A. 1973. *Nature* 242:424
Fricke, K. J., Kippenhahn, R. 1972. *Ann. Rev. Astron. Ap.* 10:45
Fricke, K. J., Stobie, R. S., Strittmatter, P. A. 1971. *MNRAS* 159:23
Fricke, K. J., Stobie, R. S., Strittmatter, P. A. 1972. *Ap. J.* 171:593
Fricke, K. J., Strittmatter, P. A. 1972. *MNRAS* 156:129
Gehrz, R. D., Woolf, N. J. 1971. *Ap. J.* 165:285
Graboske, H. C. Jr. 1973. *Explosive Nucleosynthesis*, ed. D. N. Schramm, W. D. Arnett, p. 186. Univ. Texas: Austin
Gross, P. 1971. PhD thesis. Yale Univ., New Haven, Conn.
Hallgren, E. L., Cox, J. P. 1970. *Ap. J.* 162:933
Hartwick, F. D. A. 1970. *Ap. J.* 161:845
Hartwick, F. D. A., Härm, R., Schwarzschild, M. 1968. *Ap. J.* 151:389
Hartwick, F. D. A., Vanden Berg, D. A. 1973. *Ap. J.* 185:887
Hayashi, C., Hoshi, R., Sugimoto, D. 1962. *Prog. Theor. Phys. Suppl.* 22:1
Hejlesen, P. M. 1972. *Age des Etoiles*, Proc. IAU Colloq. No. 17 (Paris-Meudon Observatory), ed. G. Cayrel de Strobel, A. M. Delplace
Herbig, G. H., Boyarchuk, A. A. 1968. *Ap. J.* 153:397
Hofmeister, E. 1967. *Z. Ap.* 65:164
Hoshi, R. 1968. *Prog. Theor. Phys.* 39:957
Humphreys, R. M. 1970. *Ap. Lett.* 6:1
Iben, I. Jr. 1967. *Ann. Rev. Astron. Ap.* 5:571
Iben, I. Jr. 1968a. *Ap. J.* 154:581
Iben, I. Jr. 1968b. *Nature* 220:143
Iben, I. Jr. 1969. *Ann. Phys.* 54:164
Iben, I. Jr. 1971. *PASP* 83:697
Iben, I. Jr. 1972. *Ap. J.* 178:433
Iben, I. Jr. 1973a. *Explosive Nucleosynthesis*, ed. D. N. Schramm, W. D. Arnett, p. 115. Univ. Texas, Austin
Iben, I. Jr. 1973b. *Ap. J.* 185:209
Iben, I. Jr. 1974. In preparation
Iben, I. Jr., Faulkner, J. 1965. AAS December meeting, Berkeley, California
Iben, I. Jr., Faulkner, J. 1968. *Ap. J.* 153:101
Iben, I. Jr., Huchra, J. P. 1971. *Astron. Ap.* 14:293

Iben, I. Jr., Rood, R. T. 1969. *Nature* 223:933

Iben, I. Jr., Rood, R. T. 1970. *Ap. J.* 159:605; 161:587

Iben, I. Jr., Rood, R. T., Strom, K. M., Strom, S. E. 1969. *Nature* 224:1006

Iben, I. Jr., Tuggle, R. S. 1972. *Ap. J.* 173:135; 178:441

Ikeuchi, S., Nakazawa, K., Marai, T., Hoshi, R., Hayashi, C. 1971. *Prog. Theor. Phys.* 46:1713

Ikeuchi, S., Nakazawa, K., Marai, T., Hoshi, R., Hayashi, C. 1972. *Prog. Theor. Phys.* 48:1870

Joss, P. C., Katz, J. I., Malone, R. C., Salpeter, E. E. 1973. *Ap. J.* 181:409

Joss, P. C., Salpeter, E. E., Ostriker, J. P. 1973. *Ap. J.* 181:429

Keeley, D. A. 1970. *Ap. J.* 161:643, 657

Kippenhahn, R., Meyer-Hofmeister, E., Thomas, H. C. 1970. *Astron. Ap.* 5:155

Kippenhahn, R., Thomas, H. C., Weigert, A. 1965. *Z. Ap.* 61:241

Kippenhahn, R., Thomas, H. C., Weigert, A. 1966. *Z. Ap.* 64:373

Kozlowski, M. 1971. *Ap. Lett.* 9:65

Kukarkin, B. V., Paranego, P. P. 1963. *Basic Astronomical Data,* ed. K. A. Strand, Chap. 18, Univ. of Chicago Press

Langer, G. E., Kraft, R. P., Anderson, K. S. 1973. *Ap. J.* In press

Larson, R. B., Starrfield, S. 1971. *Astron. Ap.* 13:190

Larsson-Leander, G. 1972. *Age des Etoiles,* Proc. IAU Colloq. No. 17 (Paris-Meudon Observatory), ed. G. Cayrel de Strobel, A. M. Delplace

Lauterborn, D. 1972. *Astron. Ap.* 19:473

Lauterborn, D., Refsdal, S., Roth, M. L. 1971. *Astron. Ap.* 13:119

Lauterborn, D., Refsdal, S., Weigert, A. 1971. *Astron. Ap.* 10:97

Ledoux, P. 1941. *Ap. J.* 94:537

Ledoux, P., Ed. 1974. *Proc. IAU Symp. No. 59,* Canberra, Australia, August 1973

Lucy, L. B. 1967. *Astron. J.* 72:813

Maeder, A. 1973. *Proc. IAU Symp. No. 59,* Canberra, Australia, ed. P. Ledoux. To be published

Maran, S. P., Brandt, J., Ed. 1974. *Proc. Conf. on the Gum Nebula,* Greenbelt, Maryland (NASA)

Massevich, A. G., Tutukov, A. V. 1973. *Proc. IAU Symp. No. 66,* Warsaw, ed. R. Taylor

Mazarakis, M., Stephens, W. 1972. *Ap. J. Lett.* 171:L97

Meyer-Hofmeister, E. 1972. *Astron. Ap.* 16:282

Misner, C., Thorne, K., Wheeler, J. 1973. *Gravitation.* San Francisco: Freeman

Newell, E. B. 1973. *Ap. J. Suppl.* 26:37

Nishida, M. 1960. *Prog. Theor. Phys. Suppl.* 23:896

Nishida, M., Sugimoto, D. 1962. *Prog. Theor. Phys.* 27:145

Nomoto, K., Sugimoto, D. 1973. *Proc. IAU Symp. No. 66,* Warsaw, ed. R. Taylor

Oosterhoff, P. Th. 1939. *Observatory* 62:104

Osaki, Y. 1963. *Publ. Astron. Soc. Jap.* 15:428

Ostriker, J. P. 1971. *Ann. Rev. Astron. Ap.* 9:353

Ostriker, J. P., Gunn, J. 1972. *Ap. Lett.* 164:L75

Paczynski, B. 1969. *Acta Astron.* 19:1

Paczynski, B. 1970. *Acta Astron.* 20:47, 195, 287

Paczynski, B. 1971a. *Acta Astron.* 21:2, 271, 417

Paczynski, B. 1971b. *Ann. Rev. Astron. Ap.* 9:183

Paczynski, B. 1972. *Acta Astron.* 22:163

Paczynski, B. 1973a. *Acta Astron.* 23:1

Paczynski, B. 1973b. *Acta Astron.* 23:191

Paczynski, B., Ergma, E. 1973. Preprint No. 33, Polish Acad. Sci., Inst. Astron.

Paczynski, B., Ziolkowski, J. 1968. *Acta Astron.* 18:255

Peebles, P. J. E. 1966. *Ap. J.* 146:542

Peimbert, M., van den Bergh, S. 1971. *Ap. J.* 167:223

Peterson, J. O. 1972. *Astron. Ap.* 19:197

Plavec, M. 1968. *Advan. Astron. Ap.* 6:201

Robertson, J. W. 1971. *Ap. J.* 164:L105; 170:353

Robertson, J. W. 1972. *Ap. J.* 173:631; 177:473

Robertson, J. W. 1973. *Ap. J.* 180:425; 185:817

Rodgers, A. W. 1970. *MNRAS* 151:133

Rood, R. T. 1970. *Ap. J.* 162:939

Rood, R. T. 1971. *Ap. J.* 169:191

Rood, R. T. 1972a. *Ap. J.* 177:681

Rood, R. T. 1972b. *Nature Phys. Sci.* 240:178

Rood, R. T. 1973. *Ap. J.* 184:815

Rood, R. T., Iben, I. Jr. 1968. *Ap. J.* 154:215

Rose, W. K. 1966. *Ap. J.* 144:1001; 146:838

Rose, W. K. 1967a. *Ap. J.* 150:193

Rose, W. K. 1967b. *Mass Loss from Stars,* ed. M. Hack, 77. Dordrecht, the Netherlands: Reidel

Rose, W. K. 1969. *Ap. J.* 155:491

Rose, W. K., Smith, R. L. 1970. *Ap. J.* 159:903

Roxburgh, I. W. 1967. *Nature* 215:838

Ruben, G. 1969. *Nauch. Inform. Acad. Nauk USSR* 14:3

Ruderman, M. 1972. *Ann. Rev. Astron. Ap.* 10:427

Salpeter, E. E. 1971. *Ann. Rev. Astron. Ap.* 9:127

Sandage, A. R. 1970. *Ap. J.* 162: 841
Sandage, A. R., Katem, B., Kristian, J. 1968. *Ap. J.* 153: L129
Sandage, A. R., Eggen, O. J. 1969. *Ap. J.* 158: 685
Sandage, A. R., Tammann, G. A. 1969. *Ap. J.* 157: 683
Sanders, R. H. 1967. *Ap. J.* 150: 971
Scalo, J. M., Ulrich, R. K. 1973. *Ap. J.* 183: 151
Schlesinger, B. M. 1969. *Ap. J.* 158: 1059
Schwarzschild, M. 1970. *Quart. J. Roy. Astron. Soc.* 11: 12
Schwarzschild, M., Härm, R. 1959. *Ap. J.* 129: 637
Schwarzschild, M., Härm, R. 1962. *Ap. J.* 136: 158
Schwarzschild, M., Härm, R. 1964. *Ap. J.* 139: 594
Schwarzschild, M., Härm, R. 1965. *Ap. J.* 142: 855
Schwarzschild, M., Härm, R. 1966. *Ap. J.* 145: 496
Schwarzschild, M., Härm, R. 1967. *Ap. J.* 150: 961
Schwarzschild, M., Härm, R. 1970. *Ap. J.* 160: 341
Schwarzschild, M., Härm, R. 1973. *Ap. J.* 184: 5
Shaviv, G., Salpeter, E. E. 1971. *Ap. J.* 165: 171
Shaviv, G., Salpeter, E. E. 1973. *Ap. J.* 184: 191
Sheldon, W. R. 1969. *Nature* 221: 650
Simoda, M. 1972. *Publ. Astron. Soc. Jap.* 24: 13
Simoda, M., Iben, I. Jr. 1968. *Ap. J.* 152: 509
Simoda, M., Iben, I. Jr. 1970. *Ap. J. Suppl. No. 183,* 22: 81
Simoda, M., Kimura, H. 1968. *Ap. J.* 151: 133
Simoda, M., Tanikawa, K. 1970. *Publ. Astron. Soc. Jap.* 22: 143
Simoda, M., Tanikawa, K. 1972. *Publ. Astron. Soc. Jap.* 24: 1
Simpson, E. E. 1971. *Ap. J.* 165: 295
Smith, R. L., Rose, W. K. 1972. *Ap. J.* 176: 395
Smith, R. L., Sackmann, I. J., Despain, K. H. 1973. *Explosive Nucleosynthesis* ed. D. N. Schramm, W. D. Arnett, p. 169. Univ. Texas, Austin
Sparks, W. M., Kutter, G. S. 1972. *Ap. J.* 175: 707
Stellingwerf, R. F. 1974. PhD thesis. Univ. Colorado, Boulder
Stothers, R. 1972. *Ap. J.* 175: 431
Stothers, R., Chin, C. W. 1968. *Ap. J.* 152: 225
Stothers, R., Chin, C. W. 1969. *Ap. J.* 158: 1039
Stothers, R., Chin, C. W. 1973. *Ap. J.* 179: 555
Strom, S. E., Cohen, J. G., Strom, K. M. 1967. *Ap. J.* 147: 1038
Strom, S. E., Strom, K. M., Rood, R. T., Iben, I. Jr. 1970. *Bull. AAS* 2: 347; *Astron. Ap.* 8: 243
Suda, K., Virgopia, N. 1966. *Ap. J.* 143: 75, 87
Sugimoto, D. 1964. *Progr. Theor. Phys.* 32: 703
Sugimoto, D. 1970. *Progr. Theor. Phys.* 44: 375, 599
Sugimoto, D. 1971. *Progr. Theor. Phys.* 45: 761
Sugimoto, D., Nomoto, D. 1973. In preparation
Sweigert, A. V. 1971. *Ap. J.* 168: 79
Sweigert, A. V. 1973. *Astron. Ap.* 24: 459
Sweigert, A. V., Demarque, P. 1972. *Astron. Ap.* 20: 445
Sweigert, A., Gross, P. 1973. Preprint
Sweigert, A., Mengel, J. G., Demarque, P. 1973. Preprint
Talbot, R. J. 1971. *Ap. J.* 163: 17; 165: 121
Taylor, R., Papaloizou, J. 1973. *Stellar Evolution and Instability,* Proc. IAU Symp. No. 59, Canberra, Australia, ed. P. Ledoux
Thomas, H. C. 1967. *Z. Ap.* 67: 420
Torres-Peimbert, S. 1971. *Bol. Observ. Tonanzintla Tacubaya* 6: 3, 15
Tsuruta, S., Cameron, A. G. W. 1969. *Ap. Space Sci.* 7: 374
Tuggle, R. S., Iben, I. Jr. 1972. *Ap. J.* 178: 455
Ulrich, R. K. 1973. *Explosive Nucleosynthesis,* ed. D. N. Schramm, W. D. Arnett, p. 139. Univ. Texas, Austin
Ulrich, R. K., Rood, R. T. 1973. *Nature Phys. Sci.* 241: 111
Ulrich, R. K., Scalo, J. M. 1972. *Ap. J.* 176: L37
Uus, U. 1970. *Nauch. Inform. Acad. Nauk USSR* 17: 3, 25, 35, 48
Uus, U. 1971. *Nauch. Inform. Acad. Nauk USSR* 20: 60, 64
van Albada, T. S., Baker, N. H. 1971. *Ap. J.* 169: 311
van Albada, T. S., Baker, N. H. 1973. *Ap. J.* 185: 477
van Altena, W. 1973. IAU General Assembly, Sydney, Australia
Varshavsky, V. I. 1972. *Nauch. Inform. Acad. Nauk USSR* 21: 25
Varshavsky, V. I., Tutukov, A. V. 1972. *Nauch. Inform. Acad. Nauk USSR* 23: 47
Varshavsky, V. I., Tutukov, A. V. 1973. *Nauch. Inform. Acad. Nauk USSR* 26: 35; 27: 73
Vidal, N. V., Shaviv, G., Koslovsky, B. Z. 1971. *Astron. Ap.* 13: 147

Vila, S. C. 1970. *Ap. J.* 162:605
von Sengbush, K. 1973. *Mitt. Ast. Ges.* No. 32
Wagoner, R., Fowler, W. A., Hoyle, F. 1967. *Ap. J.* 148:3
Wallerstein, G. 1970. *Ap. J.* 160:345
Weidemann, V. 1968. *Ann. Rev. Astron. Ap.* 6:351
Weigert, A. 1966. *Z. Ap.* 64:395
Weinberg, S. 1967. *Phys. Rev. Lett.* 19:1264
Weinberg, S. 1972. *Phys. Rev. Lett.* 27:1688
Wheeler, J. C., Barkat, Z. K., Buchler, J. R. 1970. *Ap. J. Lett.* 162:L129
Wheeler, J. C., Buchler, J. R., Barkat, Z. K. 1973. *Ap. J.* 184:897
Wickramasinghe, N. C., Donn, B. D., Stecher, T. P. 1966. *Ap. J.* 146:590
Woltjer, L. 1958. *BAN* 14:39
Wood, P. R. 1973a. Preprints
Wood, P. R. 1973b. *Stellar Evolutions and Instability,* Proc. IAU Symp. No. 59, Canberra, Australia, ed. P. Ledoux
Wood, P. R., Faulkner, D. J. 1973. *Ap. J.* 181:147
Woolf, N. 1973. *Proc. IAU Symp. No. 66,* Warsaw, ed. R. Taylor
Ziebarth, K. 1970. *Ap. J.* 162:947
Ziolkowski, J. 1972. *Acta Astron.* 22:327
Zytkow, A. 1972. *Acta Astron.* 22:103
Zytkow, A. 1973. Preprint No. 20. Polish Acad. Sci. Inst. Astron.